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Penalized Intuitionistic Fuzzy Goal Programming Method for Solving Multi-Objective Decision-Making Problems

Demmelash Mollalign Moges[1,*], Berhanu Guta Wordofa[2], Allen Rangia Mushi[3]

[1]*Department of Mathematics, Hawassa University, Hawassa, Ethiopia*
[2]*Department of mathematics, Addis Ababa University, Addis Ababa, Ethiopia*
[3]*Department of mathematics, University of Dares Salaam, Dares Salaam, Tanzania*

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Multiobjective decision-making; Intuitionistic fuzzy decision set; Goal programming; Interactive penalty function.

Research article

ABSTRACT

Many applicable problems have multi-goals that optimize simultaneously, and decision-makers set imprecise aspiration levels for each goal. Although such types of problems solved by fuzzy optimization are common in the literature, intuitionistic fuzzy optimization techniques are more efficient to handle than fuzzy and classical optimization. This research study focused on establishing a novel method by combining the penalty function method with an interactive goal programming methodology for addressing multi-objective decision-making problems in an intuitionistic fuzzy environment. One of the challenge that exists in the literature of the optimization method under an imprecise decision environment is that it is not guaranteed to generate a Pareto-optimal solution for the introduced problem. Therefore, in order to ensure the Pareto-optimality of the obtained solution, the suggested method has developed a new aggregation operator, an appropriate relaxation of the constraint set, and a well-structured extended Yager membership function. In addition, unlike other methods in the literature, the suggested method gives decision-makers the option to penalize the most unsatisfied objective function at a specific attained solution instead of starting from scratch and working their way through the problem. To illustrate the proposed method, we used a numerical example.

1. INTRODUCTION

Many real-world decision-making problems such as agricultural cropland allocation problems by Moges et al. (2023a); Basumatary and Mitra (2022), transportation problems by Tadesse et.al. (2023), water

*Corresponding author
Email address: *danieldemelashalem@gmail.com* +251913845944 https://dx.doi.org/10.4314/eajbcs.v5i2.4S

resource allocation problems by Sharma et.al. (2007), production planning decision-making problems by Khan et.al. (2021), etc are defined as multi-objective programming problems (MOPPs). Often, the input data used in these problems does not have exact numerical values because of various uncontrollable circumstances. To solve such types of MOPPs, the most appropriate and straightforward techniques are fuzzy optimization methods, which were introduced by Bellman and Zadeh (1970). Many authors (Tadesse et.al., 2023; Jameel and Radhi, 2014; Kahraman et.al., 2016) have been more committed to apply fuzzy set tools for various application issues after Bellman and Zadeh (1970) provided a model to tackle fuzzy-multi-objective programming problems (FMOPPs).

The core idea behind the widely used approach for solving FMOPPs in the literature is to transform the original FMOPPs into a crisp single-objective optimization model using aggregation operators and ranking accuracy approaches, and then solve it using classical methods (Zimmermann, 2001; Kahra-man et.al., 2016; Bellman and Zadeh, 1970). Few papers have focused on developing new mathematical models for identifying fuzzy non-dominant solutions to FMOPP in the fuzzy optimization environment. These new models were developed by applying different aggregation operators like max-min operator by Bellman and Zadeh (1970), γ-operator by Zimmermann (2001), bounded min-sum operator by Cheng et.al. (2013), and fuzzy-and operator by Singh and Yadav (2015) to convert the multi-objective functions into single-objection functions. Using the newly introduced concepts, numerous approaches and models have been developed for various theoretical and scientific fields (Bogdana and Milan, 2009; Kassa and Tsegay, 2018; Tadesse et.al., 2023). However, when uncertainty results from vagueness, inaccurate data, or intentional judgments, the modeling capabilities of fuzzy set theory are limited. Decision-makers may also experience some hesitancy as a result of imprecise information, unawareness of customers, seasonal change, etc. These kinds of factors are critical to consider while building realistic, suitable models and solving decision-making problems (Singh and Yadav, 2015; Razmi et.al, 2016; Moges and Wordofa, 2024; Kumar, 2020; Fathy et.al., 2023).

To overcome these limitations, numerous researchers suggested various fuzzy extensions that expanded the conventional fuzzy set theory concepts. Atanassov (1986) developed a novel concept called an intuitionistic fuzzy (IF) theory which successfully addresses the limitation of fuzzy theory. Since the IF set can offer degrees of acceptance, hesitancy, and rejection, it has been found to be more helpful for handling imprecision in optimization procedures than fuzzy and crisp-based models, (Mollalign et.al., 2022; Sharma et.al., 2023). Angelov (1995) proposed a new model to determine the MN[1] Pareto-optimal solution based on intuitionistic fuzzy optimization (IFO) which is a direct extension of the fuzzy optimization technique put out by Bellman and Zadeh (1970). Subsequently, numerous scholars such as (Ghosh and Kumar, 2014; Moges et al., 2023a; Rukmani and Porchelvi, 2018a; Bharati and Singh, 2014; Dey and Roy, 2015; Mollalign et.al., 2022; Moges et.al., 2023b; Bharati et.al., 2014) have been proposed different IFO methods to solve the domain of MOPPs utilizing the benefit of IFS tools.

Yager (2009) highlighted certain drawbacks of Angelov (1995)'s method for determining the optimal choice for decision-makers. He suggested a new approach by con-

[1] MN represent Membership-Nonmembership

verting the intuitionistic fuzzy decision environment (IFDE) into a fuzzy decision environment using a convex combination of non-membership and membership functions. By addressing the shortcomings of the Angelov (1995) technique, Dubey et.al. (2012) applied the Yager (2009) strategy to solve the MOPP. Garai et.al. (2015) exposes the shortcomings of the Dubey et.al. (2012) method by demonstrating how certain constraints in their model can impede the pursuit of the optimal solution or render the model impractical. They also suggest a new function by broadening the scope of non-membership and membership functions.

As with fuzzy optimization, a problem in the IFO environment can be formulated as a two-step process to be solved. The first is to convert a multi-objective function into a crisp single-objective optimization model using an appropriate aggregation operator, and then solve this model using suitable optimization technique. The majority of the existing IFO methods in the literature for resolving MOPP are based on the max-min operator, (Moges et al., 2023a; Dubey et.al., 2012; Garai et.al., 2015). However, the max-min operator is not guaranteed to generate the Pareto dominance solution, and there is always no compensation for the resulting solution. The compensatory of the developed aggregation operator has a critical role in the procedure of solving MOPPs and it affects the optimality of the resulting non-dominant solution. As a result, the MN Pareto-solution might not be the result of a model that was developed using the max-min operator approach.

The second limitation is that even if the IFO method generates a MN Pareto-optimal solution under an IFDE, there is not always a guarantee of a Pareto-optimal solution to the given MOPP due to the boundary problem. To overcome this difficulty, many researchers have proposed different approaches to fuzzy decision-making problems. But the two-phase method which used by Lu et.al. (2015); Dubois and Fortemps (1999); Wu and Guu (2001); Dubey et.al. (2012); Tsegaye et.al. (2021); Jiménez and Bilbao (2009) is the most commonly applied approach for finding a Pareto-optimal solution to MOPPs in fuzzy decision environment. A two-phase method means that in phase one, use the max-min operator technique and then use the mean-operator technique in phase two to improve the previous solution obtained by the max-min operator approach. However, Dubois and Fortemps (1999) indicated that this kind of strategy is not entirely consistent because we need to switch from the max-min operator to the mean-operator at different stages, and they suggested a multiphasons for objective functions provides upper and lower tolerances to avoid decision deadlock. Additionally, Razmi et.al (2016) point out that in situations where a certain level of satisfaction is fully achieved, there might not be a guarantee that a fuzzy efficient solution is Pareto-optimal. In order to generate the Pareto-optimal solution, Jiménez and Bilbao (2009); Razmi et.al (2016) extended the technique by Wu and Guu (2001) and developed a generic strategy based on the goal programming method. On the other-hand, Mollalign et.al. (2022) demonstrate that there is no assurance that the imprecise environment approach put forward by Razmi et.al (2016) will be the only method used to obtain the Pareto-optimal solution of MOPP when using intuitionistic fuzzy hierarchical optimization.

Furthermore, the IFO approach offers upper and lower tolerances to prevent decision stalemate when defining membership/ satisfaction and nonmembership/ dissatisfaction for objective functions. The third limitation in the literature of IFO is that to determine upper and lower tolerance, researchers use the "payoff matrix" approach without consideration of underestimation or overestimation values of the nadir point. Therefore, decision-makers may perceive the solution derived from this approach incorrectly. But the interactive approach is helpful to a decision-maker since it offers mechanisms for learning about a problem in the IFO. There are a few researchers that concern an interactive method for solving a different domain of MOPPs in an intuitionistic fuzzy decision environment, (Hanine et.al., 2021; Garai et.al., 2016). However, in the existing interactive method, decision-makers do not have the freedom to punish/ penalize the more unsatisfied objective function at each iteration when the current solution does not satisfy the DM, rather than solving the problem from scratch.

By taking into account the aforementioned limitations, the key goal of this research study is to develop a general and powerful novel method for solving the multi-objective linear decision-making problem (MOLDMP) in an IFDE using the combination of penalty function method, the interactive method, and the goal programming. Additionally, the following points are specific and basic contributions of current research-study works:

i. We formulated an IF membership and nonmembership function by finding the correct ideal and nadir point to set boundary for intuitionistic fuzzy goals (IFGs).

ii. We proposed an extended Yager-membership function that eliminates the boundary value problem.

ii. A new IF aggregation operator is developed to convert MOLDMP into a single-objective decision-making problem.

iii. Using a new IF aggregation operator, we proposed an IFG programming model to find a Pareto-optimal solution for MOLDMP.

iv. Decision-makers provide an interactive penalty function method to punish unsatisfactory objective functions at the current solution.

The rest of the paper is presented as follows: the basic concepts and terms related to the paper are presented in Section 2. Section 3 illustrates the mathematical formulation of IF-MOLDMPs and its IFG model. In Section 4, we demonstrated the newly proposed method developed based on goal programming techniques and interactive penalty function method. The general framework or algorithm of the proposed method presented in Section 5 and the numerical examples used to summarize and demonstrate the applicability of the introduced method are discussed in Section 6. The results and discussion of the study are given in Section 7. Finally, the conclusion of the present work and its future scope are given in Section 8.

2. PRELIMINARIES

2.1. Intuitionistic Fuzzy Set

Definition 2.1. (Atanassov, 1986; Fathy et.al., 2023) An **intuitionistic fuzzy set (IFS)** $\tilde{A}^I$ in a non-empty universal X is a set of ordered-triplets $\tilde{A}^I = \{< x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x) >: x \in X\}$ where, $\mu_{\tilde{A}^I} : X \to [0,1]$ represent the membership function or degree of belongingness and $\nu_{\tilde{A}^I} : X \to [0,1]$ represent non-membership function or degree of non-belongingness of the element $x \in X$ being in $\tilde{A}^I$, so that $\forall x \in X$, $0 \leq \mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) \leq 1$,. For any IFS $\tilde{A}^I$ on X, $\pi_{\tilde{A}^I}(x) = 1 - \mu_{\tilde{A}^I}(x) - \nu_{\tilde{A}^I}(x)$ is the degree of indeterminacy of $x \in \tilde{A}^I$ or $x \notin \tilde{A}^I$.

2.1.1. Operations over Intuitionistic Fuzzy Sets

Assume that $\tilde{A}^I = \{< x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x) > |x \in X\}$ and $\tilde{B}^I = \{< x, \mu_{\tilde{B}^I}(x), \nu_{\tilde{B}^I}(x) > |x \in X\}$ are any IFSs on the universal set X, (Husain et.al., 2012).

1. subset $\tilde{A}^I \subseteq \tilde{B}^I \iff \mu_{\tilde{A}^I}(x) \leq \mu_{\tilde{B}^I}(x)$ and $\nu_{\tilde{A}^I}(x) \geq \nu_{\tilde{B}^I}(x)$, $\forall x \in X$.

2. Equal Set: $\tilde{A}^I = \tilde{B}^I \iff \mu_{\tilde{A}^I}(x) = \mu_{\tilde{B}^I}(x)$ and $\nu_{\tilde{A}^I}(x) = \nu_{\tilde{B}^I}(x)$, $\forall x \in X$.
3. Complementation: $(\tilde{A}^I)^c = \{< x, \nu_{\tilde{A}^I}(x), \mu_{\tilde{A}^I}(x) > | x \in X\}$.
   To represent the minimum and maximum operator (that means: min and max operator), we use the symbols "$\wedge$" and "$\vee$" respectively.
4. Intersection: $\tilde{A}^I \cap \tilde{B}^I = \{< x, \mu_{\tilde{A}^I}(x) \wedge \mu_{\tilde{B}^I}(x), \nu_{\tilde{A}^I}(x) \vee \nu_{\tilde{B}^I}(x) > | x \in X\}$.
5. Union: $\tilde{A}^I \cup \tilde{B}^I = \{< x, \mu_{\tilde{A}^I}(x) \vee \mu_{\tilde{B}^I}(x), \nu_{\tilde{A}^I}(x) \wedge \nu_{\tilde{B}^I}(x) > | x \in X\}$.

2.2. Multi-objective programming problem

Assume that a vector-valued objective function $f : \Re^n \to \Re^k \, (f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}) \ldots, f_k(\mathbf{x}))$, and constraint functions $g : \Re^n \to \Re^m \, (g(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), g_3(\mathbf{x}), \ldots, g_m(\mathbf{x}))$ are continuously differentiable for all decision vector $\mathbf{x} = (x_1, x_2, x_3 \ldots, x_n) \in S$ is formulated in the following way: (Tsegaye et.al., 2021; Razmi et.al, 2016)

$$\begin{aligned} &\min \; f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}) \ldots, f_k(\mathbf{x}))^T \\ &\text{Subject to:} \quad \mathbf{x} \in S = \left\{ \mathbf{x} \in \Re^n \;\middle|\; \begin{array}{l} g_j(\mathbf{x}) \, (\leq, \geq, =) \, 0, \; j = 1, 2, \ldots, m \\ x_i \geq 0, \; i = 1, 2, \ldots, n. \end{array} \right\} \end{aligned} \tag{1}$$

Definition 2.2. (Li and Hu, 2009) Let a vector $\mathbf{x}^* \in S$ be a feasible solution to MOPP (1). Then

- $\mathbf{x}^*$ is **weakly Pareto-optimal** solution to MOPP (1) if there doesn't exist a new vector $\mathbf{x} \in S$ such that $f_t(\mathbf{x}) < f_t(\mathbf{x}^*)$ for all $t = 1, 2, \ldots, k$.
- $\mathbf{x}^*$ is **Pareto-optimal (efficient)** solution to MOPP (1) if there doesn't exist a new vector $\mathbf{x} \in S$ such that $f_t(\mathbf{x}) \leq f_t(\mathbf{x}^*)$ for all $t = 1, 2, \ldots, k$, and $f_t(\mathbf{x}) < f_t(\mathbf{x}^*)$ for some t.

Any Pareto-optimal solution is weakly Pareto-optimal but the converse is hold for convex optimization problem. Assume that the feasible set $S \neq \emptyset$ (or $Z = f(S) \neq \emptyset$) is compact and $f_t(\mathbf{x}), \forall t$, is continuous to unsure the Pareto-optimal is exist for MOPP (1).

3. MATHEMATICAL FORMULATION OF PROBLEM

3.1. Intuitionistic Fuzzy Multi-objective Linear Decision-Making Problem

Consider mathematical formulation of Intuitionistic Fuzzy Multi-objective Linear Decision-Making Problem (IF-MOLDMP) can be represented as: (Rukmani and Porchelvi, 2018a; Basumatary and Mitra, 2022; Garai et.al., 2015)

$$\begin{aligned} &\tilde{\max}\,(\tilde{\min})\, \tilde{f}_t^I(\mathbf{x}) = \mathbf{C}\mathbf{x} + \mathbf{d}, \qquad t = 1, 2, \ldots, k \\ &\text{Subject to:} \quad \mathbf{x} \in \tilde{S}^I = \{\mathbf{x} \in \Re^n : \tilde{g}_i^I(\mathbf{x}) \, (\tilde{\leq}, \tilde{\geq}, \approx) \, b_i, \mathbf{x} \geq 0\}, \end{aligned} \tag{2}$$

where $\mathbf{C} \in \Re^{k \times n}$, $\mathbf{d}^T \in \Re^k$ are deterministic parameters and the right-hand quantity is $b_i \in \Re$, $i = 1, 2, \ldots, m$. The IF-version of classical inequality $\leq, \geq$ and equality $=$ in the model (1) are given as $\tilde{\leq}, \tilde{\geq}$ and $\approx$, respectively. The function $\tilde{f}_t^I(\mathbf{x})$ and $\tilde{g}_i^I(\mathbf{x})$ are IF linear objective and constraints, respectively due to they have IF aspiration levels set by decision-makers (DMs). $\tilde{S}^I$ is IF feasible convex region.

3.2. Intuitionistic Fuzzy Goal Model of IF-MOLDMP

In many real-life applications, the decision-maker is allowed to specify an imprecise aspiration-level for each of the constraint and objectives function in IF-MOLDMP (2). A goal with an imprecise aspiration-level for an IF objective and constraints can be treated as an intuitionistic fuzzy goal (IFG) in the IFDE. Let $\tilde{D}^I = \{\tilde{f}_1^I(\mathbf{x}), \tilde{f}_2^I(\mathbf{x}), \dots, \tilde{f}_k^I(\mathbf{x}), \tilde{g}_1^I(\mathbf{x}), \tilde{g}_2^I(\mathbf{x}), \dots, \tilde{g}_m^I(\mathbf{x}) | \mathbf{x} \in S\}$ be a set of IFGs, having maximization (M_1) with aspiration-level $\bar{f}_t$ for $t \in M_1$, $\bar{g}_i$ for $i \in N_1$ and minimization (M_2) with aspiration-level $\underline{f}_t$ for $t \in M_2$, $\underline{g}_i$ for $i \in N_2$ for type of IFGs. Hence, to determine the crisp/ deterministic optimal solution, we need to formulate the IFG model of IF-MOLDMP (2) as: (Moges et al., 2023a; Razmi et.al, 2016)

$$\text{Determine:} \quad \mathbf{x} \in \tilde{S}^I$$
$$\text{Subject to:} \quad \begin{cases} f_t(\mathbf{x}) \tilde{\gtrsim} \bar{f}_t,\ g_i(\mathbf{x}) \tilde{\gtrsim} \bar{g}_i \text{ for } t \in M_1, i \in N_1 \\ f_t(\mathbf{x}) \tilde{\lesssim} \underline{f}_t,\ g_i(\mathbf{x}) \tilde{\lesssim} \underline{g}_i \text{ for } t \in M_2, i \in N_2 \end{cases} \tag{3}$$

In order to find a Pareto-optimal solution for IF-MOLDMP (2) in the IFDE, we need to determine a solution that simultaneously maximizes the level of satisfaction/ membership $\mu_t(f_t(\mathbf{x}))$, $\mu_i(g_i(\mathbf{x}))$ and minimizes the level of dissatisfaction/ non-membership $\nu_t(f_t(\mathbf{x}))$, $\nu_i(g_i(\mathbf{x}))$ of the IFGs under the given feasible region, $\mathbf{x} \in S$, according to Angelov (1995) stated. Based on this idea, the Pareto-optimal and MN Pareto-optimal solution of IF-MOLDMP (2) are defined as follows:

Definition 3.1. (Razmi et.al, 2016; Tsegaye et.al., 2021) A solution vector $\mathbf{x}^* \in \tilde{S}^I$ is said to be a Pareto-optimal to IF-MOLDMP (2) if there is no new-vector $\mathbf{x} \in \tilde{S}^I$ such that

$$\tilde{f}_t^I(\mathbf{x}) \tilde{\leq} \tilde{f}_t^I(\mathbf{x}^*) \wedge \mu_i(g_i(\mathbf{x})) \geq \mu_i(g_i(\mathbf{x}^*)) \wedge \nu_i(g_i(\mathbf{x})) \leq \nu_i(g_i(\mathbf{x})^*), \forall t = 1, 2, \dots, k, \forall i = 1, 2, \dots, m,$$
$$\tilde{f}_t^I(\mathbf{x}) \tilde{<} \tilde{f}_t^I(\mathbf{x}^*) \vee \mu_i(g_i(\mathbf{x})) > \mu_i(g_i(\mathbf{x}^*)) \vee \nu_i(g_i(\mathbf{x})) < \nu_i(g_i(\mathbf{x})^*), \text{ for some } t \in \{1, 2, \dots, k\}, i \in \{1, 2, \dots, m\}.$$

Definition 3.2. (Jafarian et.al., 2018; Razmi et.al, 2016) A solution vector $\mathbf{x}^* \in \tilde{S}^I$ is said to be a MN Pareto-optimal solution to IF-MOLDMP (2) if there doesn't exists another vector $\mathbf{x} \in \tilde{S}^I$ such that $\mu_t(f_t(\mathbf{x})) \geq \mu_t(f_t(\mathbf{x}^*)) \wedge \nu_t(f_t(\mathbf{x})) \leq \nu_t(f_t(\mathbf{x})^*) \wedge \mu_i(g_i(\mathbf{x})) \geq \mu_i(g_i(\mathbf{x}^*)) \wedge \nu_i(g_i(\mathbf{x})) \leq \nu_i(g_i(\mathbf{x})^*), \forall t = 1, 2, \dots, k, \forall i = 1, 2, \dots, m$, and strictly inequality holds for some t or i.

4. PROPOSED SOLUTION METHOD

4.1. Extended Yager-membership function in the IFDE

In the IFDE, the nadir and ideal points are useful for estimating the range of degrees of satisfying (membership) and rejection (non-membership) for the objective function in the IF-MOLDMP (2). To determine nadir and ideal points, we need to find the optimal solution for each objective function $f_t(\mathbf{x})$ subject to the given set of constraints. That means solving the following problem independently:

$$\min f_t(\mathbf{x}) \quad \text{Subject to} \quad \mathbf{x} \in S \quad \text{for} \quad t = 1, 2, 3, \dots, k \tag{4}$$

The solution obtained from model (4) we call as $\mathbf{x}_t^B$ is best solution, $\underline{Z}_t^B = f_t(\mathbf{x}_t^B)$ is objective value or aspiration-level for each t. $Z^{id} = (\underline{Z}_1^B, \underline{Z}_2^B, \dots, \underline{Z}_k^B)$ is an ideal point and $Z^{nad} = (\bar{Z}_1, \bar{Z}_2, \dots, \bar{Z}_k)$ is nadir point of IF-MOLDMP (2) where, $\bar{Z}_t = \max_{\mathbf{x} \in P^*} f_t(\mathbf{x})$, P^* is Pareto-optimal set. Since $P^* \subset S$, we have $\max_{\mathbf{x} \in S} f_t(\mathbf{x}) \geq \bar{Z}_t, \forall t = 1, 2, 3, \dots k$.

Due to the difficulties in finding the nadir point, some scholars suggested a heuristic-approach called "Payoff matrix", but the result obtained by this approach may be an underestimation or overestimation of the nadir point, Isermann and Steuer (1997). The best solution $\bar{x}_t^B$ is used to construct the payoff matrix as follows:

$$\begin{bmatrix} & f_1(\mathbf{x}) & f_2(\mathbf{x}) & \dots & f_k(\mathbf{x}) \\ \mathbf{x}_1^B & f_1(\mathbf{x}_1^B) & f_2(\mathbf{x}_1^B) & \dots & f_k(\mathbf{x}_1^B) \\ \mathbf{x}_2^B & f_1(\mathbf{x}_2^B) & f_2(\mathbf{x}_2^B) & \dots & f_k(\mathbf{x}_2^B) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_k^B & f_1(\mathbf{x}_k^B) & f_2(\mathbf{x}_k^B) & \dots & f_k(\mathbf{x}_k^B) \end{bmatrix} \quad (5)$$

If the best solutions $\mathbf{x}_t^B$ are unique for all $t = 1, 2, 3, \dots, k$, then the payoff matrix approach will never overestimate the nadir point Z^{nad}, (Isermann and Steuer, 1997). However, if $\mathbf{x}_t^B$ not unique for at least one t, $max\{f_t(\mathbf{x}_1^B), f_t(\mathbf{x}_2^B), \dots, f_t(\mathbf{x}_k^B)\}$ may overestimate the nadir point. When a component of the nadir point for an optimization problem is underestimated or overestimated in an IFDE, it may result in unnecessary information about the Pareto-optimal solution, and decision-makers use a wide-range.

If the model (4) produces more than one best solution for any $s \in \{1, 2, \dots, k\}$, then underestimated or overestimated solutions of nadir point exist. To estimate the correct nadir point, we use two-steps in the proposed method. In step one, we solve the following optimization problem for each s, using $\sigma = \frac{1}{k} > 0$.

$$\min \frac{f_s(\mathbf{x})}{\underline{Z}_s^B} + \sigma \sum_{t \neq s}^{k} \frac{f_t(\mathbf{x}_t)}{\underline{Z}_t^B} \qquad \text{Subject to:} \quad \mathbf{x} \in S \quad (6)$$

Let $\mathbf{x}_s^{BB}$ be a solution obtained from the model (6), and construct the Payoff-matrix (5) using this solution instead of $\mathbf{x}_s^B$. Using this two-step approach, we find upper and lower-bounds for the membership and non-membership functions of the objective function, which are used as degrees of rejection and acceptance of each IFGs.

In the case of minimization problem, $U_t^\mu = max\{f_t(\mathbf{x}_1^B), f_t(\mathbf{x}_2^B), \dots, f_t(\mathbf{x}_k^B)\}$ and $L_t^\mu = min\{f_t(\mathbf{x}_1^B), f_t(\mathbf{x}_2^B), \dots, f_t(\mathbf{x}_k^B)\}$ considered as upper and lower-bound of membership functions, respectively. Similarly, $U_t^\nu = U_t^\mu$ and $L_t^\nu = L_t^\mu + \epsilon_t(U_t^\mu - L_t^\mu)$ where, $0 < \epsilon_t < 1$ for each t are upper and lower-bound of non-membership functions, respectively (Bharati and Singh, 2014; Moges et.al., 2023b).

Lemma 4.1. The solution obtained from the model (6) never generates an overestimation of Z^{nad}, (Ehrgott, 2000; Isermann and Steuer, 1997).

For minimization problem, membership $(\mu_t(f_t(\mathbf{x})))$ and non-membership $(\nu_t(f_t(\mathbf{x})))$ functions can be defined according to Eqs. 7 and Eqs. 8 respectively.

$$\mu_t(f_t(\mathbf{x})) = \begin{cases} 1 & \text{if } f_t(\mathbf{x}) \le L_t^\mu \\ \frac{U_t^\mu - f_t(\mathbf{x})}{U_t^\mu - L_t^\mu} & \text{if } L_t^\mu < f_t(\mathbf{x}) \le U_t^\mu \\ 0 & \text{if } f_t(\mathbf{x}) > U_t^\mu \end{cases} \quad (7)$$

$$\nu_t(f_t(\mathbf{x})) = \begin{cases} 0 & \text{if } f_t(\mathbf{x}) < L_t^\nu \\ \frac{f_t(\mathbf{x}) - L_t^\nu}{U_t^\nu - L_t^\nu} & \text{if } L_t^\nu \le f_t(\mathbf{x}) \le U_t^\nu \\ 1 & \text{if } f_t(\mathbf{x}) > U_t^\nu \end{cases} \quad (8)$$

Now, to overcome the limitation of the Angelov (1995) model, we first resolve the indeterminacy factors independently for each IFG using the Yager (2009) approach. Therefore, using the membership function $\mu_t(f_t(\mathbf{x}))$ defined in Eqs.7 and the non-membership function $\nu_t(f_t(\mathbf{x}))$ defined in Eqs.8, the new extended Yager-membership function is defined as follows: for any $\lambda \in [0, 1]$.

$$I_t^\lambda(f_t(\mathbf{x})) = (1 - \lambda)\mu_t(f_t(\mathbf{x})) + \lambda(1 - \nu_t(f_t(\mathbf{x}))) \quad \text{for each } t = 1, 2, \dots k. \quad (9)$$

Without knowing any other details regarding the DM's attitude during this study, we arrived at $\lambda = \frac{1}{2}$ in our discussion that follows. The piece-wise linear Yager-membership function for the minimization problem shown in Figure 1(a): and defined as:

$$I_t^\lambda(f_t(\mathbf{x})) = \begin{cases} 1 & \text{if } f_t(\mathbf{x}) \le L_t^\mu \\ 1 - \frac{1}{2}\epsilon_t(\frac{f_t(\mathbf{x})-L_t^\mu}{L_t^\nu - L_t^\mu}) & \text{if } L_t^\mu \le f_t(\mathbf{x}) < L_t^\nu \\ \frac{1}{2}(2-\epsilon_t)\frac{U_t^\mu - f_t(\mathbf{x})}{U_t^\mu - L_t^\nu} & \text{if } L_t^\nu \le f_t(\mathbf{x}) < U_t^\mu \\ 0 & \text{if } f_t(\mathbf{x}) \ge U_t^\mu \end{cases} \tag{10}$$

Now, we need to expand the range of function $I_t^\lambda(f_t(\mathbf{x})) = 1$, for $f_t(\mathbf{x}) \le L_t^\mu$ to $I_t^\lambda(f_t(\mathbf{x})) > 1$, for $f_t(\mathbf{x}) < L_t^\mu$ to overcome the limitation of Dubey et.al. (2012) model.

Suppose $L_t^{min} = min\{L_1^\mu, L_2^\mu, \ldots, L_k^\mu\} \pm L_t^\mu$ and $U_t^{max} = max\{U_1^\mu, U_2^\mu, \ldots, U_k^\mu\} \pm U_t^\mu$ such that $L_t^{min} < L_t^\mu \le f_t(\mathbf{x}) \le U_t^\mu$ for each $t = 1, 2, \ldots, k$. To extend the Yager-membership function, choose two positive numbers, $b_1, b_2 > 0$, and relax the range from $[0, 1]$ to $[-b_2, 1 + b_1]$. Therefore, the extended Yager-membership function for the minimization problem developed as shown in Figure 1(b): and defined in the following way:

$$\eta_t(f_t(\mathbf{x})) = \begin{cases} \frac{b_1(L_t^\mu - f_t(\mathbf{x}))}{L_t^\mu - L_t^{min}} + 1 & \text{if } L_t^{min} < f_t(\mathbf{x}) < L_t^\mu \\ 1 - \frac{1}{2}\epsilon_t(\frac{f_t(\mathbf{x})-L_t^\mu}{L_t^\nu - L_t^\mu}) & \text{if } L_t^\mu \le f_t(\mathbf{x}) < L_t^\nu \\ \frac{1}{2}(2-\epsilon_t)\frac{U_t^\mu - f_t(\mathbf{x})}{U_t^\mu - L_t^\nu} & \text{if } L_t^\nu \le f_t(\mathbf{x}) < U_t^\mu \\ \frac{b_2(f_t(\mathbf{x}) - U_t^\mu)}{U_t^\mu - U_t^{max}} & \text{if } U_t^\mu \le f_t(\mathbf{x}) \le U_t^{max} \end{cases} \tag{11}$$

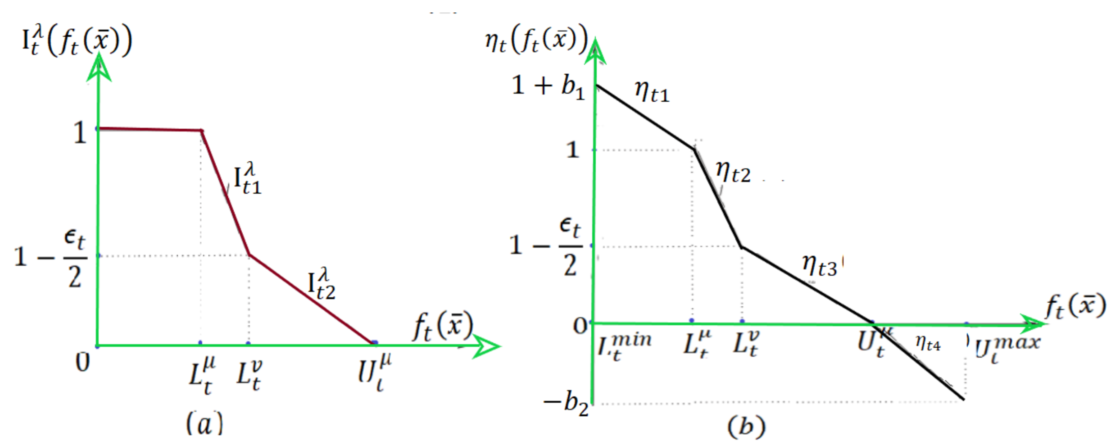

Figure 1: Yager-membership function $I_t^\lambda(f_t(\mathbf{x}))$, (Aggarwal et.al., 2019) and Extended Yager-membership function $\eta_t(f_t(\mathbf{x}))$, (Garai et.al., 2016) for the minimization problem

The piecewise linear extended Yager-membership function Eqs. (11) has the following properties:

- Over the interval containing all possible values of the objective function, it is a strictly monotonic function.
- It permits alternate point orderings with Yager-membership function values outside of $[0, 1]$.
- It is equivalent to the Yager-membership function $I_t^\lambda(f_t(\mathbf{x}))$ Eqs. (10) on $[L_t^\mu, U_t^\mu]$ for each t.
- If $f_t(\mathbf{x}) < L_t^\mu$, then $\eta_t(f_t(\mathbf{x})) \ge 1$, and if $f_t(\mathbf{x}) > U_t^\mu$, then $\eta_t(f_t(\mathbf{x})) \le 0$.
- It shows the level of satisfaction of the decision-maker for every objective function.

Additionally, we modified these piecewise linear extended Yager-membership functions $\eta_t(f_t(\mathbf{x}))$ as follows: based on the method suggested by Wu and Guu (2001) to solve MOPP (1) under an IFDE without introducing extra binary variables.

$$\eta_t(f_t(\mathbf{x})) = \eta_t(a_t^1) + s_t^1(f_t(\mathbf{x}) - a_t^1) + \sum_{l=2}^{m-1} \frac{(s_t^l - s_t^{l-1})}{2}(|f_t(\mathbf{x}) - a_t^l| + f_t(\mathbf{x}) - a_t^l) \quad (12)$$

where, a_t^l, $l = 1, 2, 3, \ldots, m$ are the breaking/ jumping points of $\eta_t(f_t(\mathbf{x})), t = 1, \ldots, k$, the slope of line segment between a_t^l and a_t^{l+1} is given by $s_t^l = \frac{\eta_t(a_t^{l+1}) - \eta_t(a_t^l)}{a_t^{l+1} - a_t^l}$ for $l = 1, \ldots, m-1; t = 1, 2, \ldots, k$.

4.2. Develop a new intuitionistic fuzzy (IF) aggregation operator

A number of logical operators are available in the IFDE, enabling the combination of numerous objectives into a single objective. We employed the convex combination of the arithmetic mean-operator and the max-min operator on the extended Yager-membership functions $\eta_t(f_t(\mathbf{x}))$ Eqs. (11) in order to create a novel operator in the IFDE. Therefore, the new IF aggregation operator is formulated as:

$$\eta_{\tilde{D}^I}(\mathbf{x}) = \delta \, min_{t=1}^{k} \eta_t(f_t(\mathbf{x})) + (1-\delta)\frac{1}{k}\sum_{t=1}^{k} \eta_t(f_t(\mathbf{x})), \quad (13)$$

where $0 < \delta < 1$ represent the degree of preference of DMs. Using this novel IF aggregation operator and decision-makers' preference value $\delta \in [0, 1]$, the IF-MOLDMP (2) is transformed into a crisp single-objective optimization problem to determine the Pareto-optimal solution based on Eqs.(14).

$$\max_{\mathbf{x}\in S} \eta_{\tilde{D}^I}(\mathbf{x}) \quad (14)$$

Therefore, using a non-negative variable q_t^l, the extended Yager-membership function defined in Eqs.(12), and the IF aggregation operator defined in (13), the equivalent deterministic single-objective optimization problem of model (14) is formulated as follows:

Model I:

$$\max \delta\alpha_0 + (1-\delta)\sum_{t=1}^{k} \alpha_t$$

$$\text{Subject to: } \left\{ \begin{array}{l} \alpha_0 + \alpha_t \leq \eta_t(f_t(\mathbf{x})) \quad \text{for } t = 1, 2, \ldots, k. \\ \eta_t(f_t(\mathbf{x})) = \eta_t(a_t^1) + s_t^1(f_t(\mathbf{x}) - a_t^1) + \cdots + (s_t^{m-1} - s_t^{m-2})(f_t(\mathbf{x}) - a_t^{m-1} + q_t^{m-2}) \\ f_t(\mathbf{x}) + q_t^{l-2} \geq a_t^{l-1} \text{ for } l = 3, 4, \ldots, m \\ \alpha_0, \alpha_t, q_t^{l-2} \geq 0 \quad \text{for } t = 1, 2, \ldots, k; l = 3, 4, \ldots, m. \\ \quad \mathbf{x} \in S \end{array} \right\} \quad (15)$$

4.3. The intuitionistic fuzzy goal programming method

The higher-value of $\eta_t(f_t(\mathbf{x}))$ is regarded as the best acceptable value for DMs in the IFDE. As a result, DMs must minimize the under-achievement (negative-deviational) variable d_t^- by assigning weight w_t^- to each objective function t in the goal programming approach $\eta_t(f_t(\mathbf{x})) + d_t^- \geq 1$. Therefore, the intuitionistic fuzzy goal programming (IFGP) model of IF-MOLDMP (2) is formulated as follows:

Model II:

$$\min \sum_{t=1}^{k} w_t^- d_t^-$$
$$\text{Subject to:} \quad \left\{ \begin{array}{l} \eta_t(f_t(\mathbf{x})) + d_t^- \geq 1 \text{ for } t = 1, 2, 3, \ldots, k \\ \mathbf{x} \in S \\ d_t^- \geq 0, \ \sum_{t=1}^{k} w_t^- = 1, w_t^- > 0 \end{array} \right\} \tag{16}$$

4.4. Interactive Penalty Function Method (IPFM)

In an intuitionistic fuzzy decision environment (IFDE), the interactive approach is very effective at solving IF-MOLDMP (2). In this process, the algorithm generates an initial solution before consulting the DM and obtaining a new solution(s) if the DM is dissatisfied with the current one. Let $\mathbf{x}^*$ be the optimal solution of model I (15) or model II (16). Assume that the existing solution $\mathbf{x}^*$ does not satisfy the DMs.

To discuss how the penalty function method is applied in the interactive method once DMs update the problem for some objective functions, we considered the following Figure 2 that shows the level of satisfaction and dissatisfaction by DMs. Furthermore, the Figure 2 demonstrates how to determine a solution from a large space by relaxing the constraint set. The green region in the Figure 2 represents DMs who are completely satisfied with the current solution, while the red region represents DMs who are dissatisfied, and the row depicts how the algorithm generates the best solution from a large space.

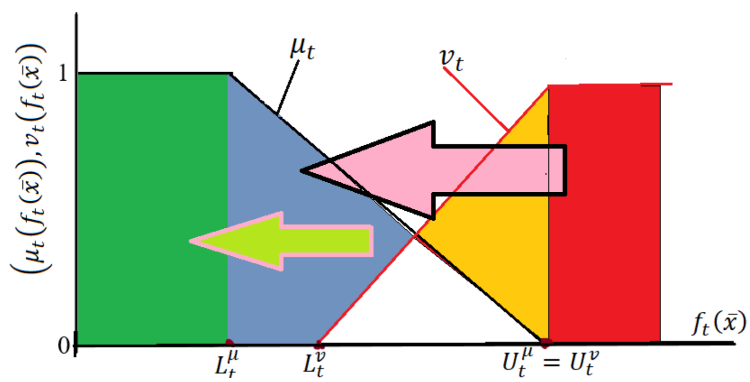

Figure 2: How penalty function method applied on minimization type problem

Definition 4.1. A vector $\mathbf{x} \in S$ is DM-feasible solution if $\forall t \in 1, 2, \ldots, k, \eta^\lambda_{\tilde{D}^I}(f_t(\mathbf{x})) - \eta^\lambda_{\tilde{D}^I}(f_t(\mathbf{x}^*)) \geq \gamma_t$, for nonnegative variable γ_t. Where, $\mathbf{x}^*$ is optimal solution to either model I (15) or model II (16).

Let the set of DM-feasible solution be denoted by S^{DM} and defined as $S^{DM} = \{\mathbf{x} \in S | \eta^\lambda_{\tilde{D}^I}(f_t(\mathbf{x})) - \eta^\lambda_{\tilde{D}^I}(f_t(\mathbf{x}^*)) \geq \gamma_t, \forall t\}$.

To find a preferable Pareto-optimal solution based on $\mathbf{x}^*$ to IF-MOLDMP (2), we solve the following optimization problem:

$$\max_{\mathbf{x}\in S^{DM},\gamma_t\geq 0} \sum_{t=1}^{j} \gamma_t \tag{17}$$

Now choose $\mathbf{x} \in S$ such that $\eta_t^{\lambda}(f_t(\mathbf{x})) - \eta_t^{\lambda}(f_t(\mathbf{x}^*)) < \gamma_t^2$ (i.e.,$\mathbf{x} \notin S^{DM}$). The non-linear penalty function is defined as $P(\mathbf{x}) = \sum_{t=1}^{j}[max\{0, \eta_{\tilde{D}^I}^{\lambda}(f_t(\mathbf{x}^*)) - \eta_{\tilde{D}^I}^{\lambda}(f_t(\mathbf{x})) + \gamma_t^2\}]^2$ Meng et.al. (2011); Jameel and Radhi (2014) and satisfies:

$$P(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in S^{DM} \\ > 0 & \text{if } \mathbf{x} \notin S^{DM} \end{cases} \tag{18}$$

Now, using the penalty function, we can transform it into unconstrained optimization, where the objective function is defined as:
$F(\mathbf{x}, c_i) = \sum_{t=1}^{j} \gamma_t + c_i(\sum_{t=1}^{j}[max\{0, \eta_t^{\lambda}(f_t(\mathbf{x}^*)) - \eta_t^{\lambda}(f_t(\mathbf{x})) + \gamma_t^2\}]^2)$
Based on the penalty parameter $c_i > 0$, we formulate the penalty optimization problem as follows:

$$\begin{aligned} &(P_{\lambda,c_i}) \quad max\, F(\mathbf{x}, c_i) \\ &\text{Subject to} \quad \mathbf{x} \in R^n, \end{aligned} \tag{19}$$

The main benefit of this model is that, instead of starting from scratch, it will use a wide domain to improve the existing solution and attempt to discover a solution that satisfies DMs by ignoring the non-update membership and non-membership functions.

IPFM Algorithm:

Initial Step:

- Using current solution $\mathbf{x}^*$ and $\gamma_{t_i} = \eta_{\tilde{D}^I}^{\lambda}(\frac{U_t^{\mu} - L_t^{\mu}}{2})$, choose initial solution $\mathbf{x}_1 \in S$ such that $\eta_t^{\lambda}(f_t(\mathbf{x}_1)) - \eta_t^{\lambda}(f_t(\mathbf{x}^*)) < \gamma_{t_i}^2$. That means: $\mathbf{x}_1 \notin S^{DM}$.
- Choose $\lambda = \frac{1}{2}, c_1 > 0, \beta > 1$ and set i=1

Main Step:

Step I. Using $\mathbf{x}_i$ and γ_{t_i} solve the problem $max\, F(\mathbf{x}, c_i)$ Subject to $\mathbf{x} \in R^n$ and let call $\mathbf{x}_{i+1}$ be an optimal solution.

Step II. If $\mathbf{x}_{i+1}$ be DM feasible solution, then stop $\hat{x} = \mathbf{x}_{i+1}$ and the corresponding decision vector $\hat{\gamma}$ are optimal solutions to model (17). Otherwise, put $c_{i+1} = \beta c_i$, $i = i + 1$ and go to step I.

Theorem 4.1 (Optimality test). Let $\hat{x}$ and $\hat{\gamma}$ are optimal solutions to model (17). Then

i. If $\gamma_t = 0, \forall\, t = 1, \ldots k$, then the Pareto optimal solution to IF-MOLDMP (2) in an IFDE is $\mathbf{x}^*$.

ii. In an IFDE, $\mathbf{x}^*$ is not Pareto's optimal solution to IF-MOLDMP (2) if at least one $\gamma_t > 0$. Instead of $\mathbf{x}^*$, Pareto's optimal solution to IF-MOLDMP (2) is $\hat{x}$.

(Analogous theorem proofs can be found in literature for instance see Garai et.al. (2015, 2016))

5. ALGORITHM FOR PENALIZED IFGP METHOD

Based on the idea discussed above, we proposed a general framework or algorithm for finding Pareto-optimal solutions for IF-MOLDMP (2) in an IFDE. The proposed algorithm's steps are as follows:

Step 1: Solve each objective function independently under the constraint set. That means: solve model (4).

Step 2: If the solution $\mathbf{x}_t^B$ of model (4) is unique for each t, then go to step 3. Otherwise, solve model (6) and use its solution $\mathbf{x}_t^{BB}$ instead of $\mathbf{x}_t^B$, then go to step 3.

Step 3: Construct the payoff matrix (see Eqs. 5) and find the upper and lower-tolerances of membership and non-membership functions.

Step 4: Formulate the extended Yager-membership function in an intuitionistic fuzzy decision environment. See Eqs. (11).

Step 5: Solve either model I (15) or model II (16).
If the decision-maker is satisfied with the current solution $\mathbf{x}^*$, then stop, and the current solution $\mathbf{x}^*$ is the Pareto optimal solution to IF-MOLDMP (2). Otherwise, go to step 6.

Step 6: Ask the decision-maker to change the membership and nonmembership functions for $j \leq k$ objective functions, then go to step 7.

Step 7: Solve model 17 with the above-mentioned algorithm for interactive penalty function method to get solutions $\hat{x}$ and γ_t. Then there are the following scenarios:

Case-I If $\gamma_t = 0, \forall t = 1, \ldots k$, decision-makers must adjust either δ for model I and weight of objective function for model II or λ, then go to step 5.

Case-II In an IFDE, if at least one $\gamma_t > 0$, then Pareto's optimal solution to IF-MOLDMP (2) is $\hat{x}$.

6. NUMERICAL EXAMPLE

Consider the following intuitionistic fuzzy multi-objective linear decision-making problem (IF-MOLDMP) that is given in an intuitionistic fuzzy decision environment:

$$
\begin{aligned}
&\tilde{max}\ f_1(\mathbf{x}) = -x_1 + 2x_2 \\
&\tilde{max}\ f_2(\mathbf{x}) = 2x_1 + x_2 \\
\text{Subject to:} \quad & \mathbf{x} \in S = \left\{ \mathbf{x} \in R^2 \;\middle|\; \begin{array}{l} -x_1 + 3x_2 \leq 21, 4x_1 + 3x_2 \leq 45, \\ x_1 + 3x_2 \leq 27, 3x_1 + x_2 \leq 30, \\ x_1, x_2 \geq 0. \end{array} \right\}
\end{aligned}
\tag{20}
$$

Following the proposed solution method discussed so far, we solve the given IF-MOLDMP (20) step by steps:

- Individual Solution: $\mathbf{x}_1^B = (0, 7), \mathbf{x}_2^B = (9, 3)$, and Ideal point: $(Z_1^B, Z_2^B) = (14, 21)$
- Using Payoff matrix the upper and lower bounds are: $U_1^\mu = 14, L_1^\mu = -3, L_1^{min} = -6, U_1^{max} = 27$ and $U_2^\mu = 21, L_2^\mu = 7, L_2^{min} = -10, U_2^{max} = 32$ using a tolerances $\epsilon_1 = 0.4$, $\epsilon_2 = 0.3$.
- The brake points are:
$a_1^1 = -6, a_1^2 = -3, a_1^3 = 7, a_1^4 = 14, a_1^5 = 27$ and its corresponding values are $\eta_1(a_1^1) = -b_2, \eta_1(a_1^2) = 0, \eta_1(a_1^3) = 0.8, \eta_1(a_1^4) = 1, \eta_1(a_1^5) = 1 + b_1$ for objective function $f_1(\mathbf{x})$.
$a_2^1 = -10, a_2^2 = 7, a_2^3 = 17, a_2^4 = 21, a_1^5 = 32$ and its corresponding values are $\eta_2(a_2^1) = -b_2, \eta_2(a_2^2) = 0, \eta_2(a_2^3) = 0.85, \eta_2(a_2^4) = 1, \eta_2(a_2^5) = 1 + b_1$ for objective function $f_2(\mathbf{x})$.
- The extended Yager-membership functions are:
$$
\begin{aligned}
\eta_1(f_1(\mathbf{x})) &= -0.0286x_1 + 0.057x_2 - 0.59q_1^1 \\
&\quad - 0.0514q_1^2 + 0.61 \text{ and} \\
\eta_2(f_2(\mathbf{x})) &= 0.074x_1 + 0.037x_2 - 0.033q_2^1 \\
&\quad - 0.048q_2^2 + 0.227
\end{aligned}
$$
- Based on the proposed model (15), we formulated IF-MOLDMP (20) as single-objective optimization model:

Model I: Assign the value of $\delta \in [0,1]$

$$\max \delta\alpha_0 + (1-\delta)\alpha_1 + (1-\delta)\alpha_2$$
$$\text{Subject to} \begin{cases} \alpha_0 + \alpha_1 + 0.0286x_1 - 0.057x_2 + 0.59q_1^1 + 0.0514q_1^2 \le 0.61 \\ \alpha_0 + \alpha_2 - 0.074x_1 - 0.037x_2 + 0.033q_2^1 + 0.048q_2^2 \le 0.227 \\ x_1 - 2x_2 - q_1^1 \le 3,\ x_1 - 2x_2 - q_1^2 \le -7 \\ -2x_1 - x_2 - q_2^1 \le -7,\ -2x_1 - x_2 - q_2^2 \le -17 \\ -x_1 + 3x_2 \le 21, 4x_1 + 3x_2 \le 45, x_1 + 3x_2 \le 27, 3x_1 + x_2 \le 30 \\ x_1, x_2, \alpha_0, \alpha_1, \alpha_2, q_1^1, q_1^2, q_2^1, q_2^2 \ge 0. \end{cases} \tag{21}$$

- In a similar way, using the proposed model (16), we converted the given IF-MOLDMP (20) into the following single-objective optimization model:

Model II: Assign the relative important of each objective function w_t^-

$$\min w_1^- d_1^- + w_2^- d_2^-$$
$$\text{Subject to:} \begin{cases} 0.0286x_1 - 0.057x_2 + 0.59q_1^1 + 0.0514q_1^2 - d_1^- \le -0.39 \\ -0.074x_1 - 0.037x_2 + 0.033q_2^1 + 0.048q_2^2 - d_2^- \le -0.773 \\ x_1 - 2x_2 - q_1^1 \le 3,\ x_1 - 2x_2 - q_1^2 \le -7 \\ -2x_1 - x_2 - q_2^1 \le -7,\ -2x_1 - x_2 - q_2^2 \le -17 \\ -x_1 + 3x_2 \le 21, 4x_1 + 3x_2 \le 45, x_1 + 3x_2 \le 27, 3x_1 + x_2 \le 30 \\ x_1, x_2, d_1^-, d_2^-, q_1^1, q_1^2, q_2^1, q_2^2 \ge 0. \end{cases} \tag{22}$$

Now, using MATLAB-R2023 software to solve model I (21) and model II (22), we obtained the Pareto-optimal solution $\mathbf{x_1 = 6, x_2 = 7, f_{val} = 1.1311}$, $\mathbf{f_1(x^*) = 8}$, and $\mathbf{f_2(x^*) = 19}$ for the given IF-MOLDMP (20) and shown in Table 1.

Table 1: Results for model I (21) and model II (22) in different cases

	δ, w_t^-	Optimal Solution	Optimal Value
Model I	$\delta = 0.36$	$\alpha_0 = 0, \alpha_1 = 0.83739, \alpha_2 = 0.929$, $\mathbf{x_1 = 6, x_2 = 7, f_{val} = 1.1311}$	$\mathbf{f_1(x^*) = 8}$, $\mathbf{f_2(x^*) = 19}$
	$\delta = 0.5$	$\alpha_0 = 0, \alpha_1 = 0.8374, \alpha_2 = 0.93$, $x_1 = 5.991, x_2 = 7, f_{val} = 0.8837$	$f_1(\mathbf{x}^*) = 8.009$, $f_2(\mathbf{x}^*) = 18.982$
	$\delta = 0.8$	$\alpha_0 = 0.8777, \alpha_1 = 0, \alpha_2 = 0$, $x_1 = 5.152, x_2 = 7.282, f_{val} = 0.7022$	$f_1(\mathbf{x}^*) = 9.412$, $f_2(\mathbf{x}^*) = 17.586$
Model II	$w_1^- = 0.6, w_2^- = 0.4$	$d_1^- = 0.10548, d_2^- = 0.144$, $x_1 = 4.800, x_2 = 7.399, f_{val} = 0.12089$	$f_1(\mathbf{x}^*) = 9.998$, $f_2(\mathbf{x}^*) = 16.882$
	$w_1^- = 0.5, w_2^- = 0.5$	$d_1^- = 0.1626, d_2^- = 0.0700$, $x_1 = 6.00, x_2 = 6.999, f_{val} = 0.1163$	$f_1(\mathbf{x}^*) = 7.998$, $f_2(\mathbf{x}^*) = 18.999$
	$w_1^- = 0.3, w_2^- = 0.7$	$d_1^- = 0.1626, d_2^- = 0.0700$, $\mathbf{x_1 = 6, x_2 = 7.00, f_{val} = 0.0978}$	$\mathbf{f_1(x^*) = 8}$, $\mathbf{f_2(x^*) = 19}$

7. RESULTS AND DISCUSSION

For different values of $\delta \in (0.35, 0.85)$ and $w_1^-, w_2^- \geq 0$ with $w_1^- + w_2^- = 1$, the comparative work of the resulted efficient solution is given in Table 1. Let the optimal or objective value of model I (21) and model II (22) be presented by f_{val}. As shown in Table 1, when the value of w_2^- (relative-weight of $f_2(\mathbf{x})$) increases and the value of δ (degree of compensation), w_1^- (relative-weight of $f_1(x)$) decreases, the preferable optimal value f_{val} is obtained. Thus, the optimal solution of both model I (21) and model II (22) is identical, i.e., $\mathbf{x_1 = 6, x_2 = 7, f_{val} = 1.1311}$, which is the candidate Pareto-optimal solution of IF-MOLDMP (20). Now to test the Pareto-optimality, we need to formulate the following single-objective problem (23) based on Eqs. (17) and solve it:

$$\max \quad \gamma_1 + \gamma_2$$
$$\text{Subject to:} \quad \left\{ \begin{array}{l} -0.0286x_1 + 0.057x_2 - 0.59q_1^1 - 0.0514q_1^2 - \gamma_1 \geq 0.2274 \\ 0.074x_1 + 0.037x_2 - 0.033q_2^1 - 0.048q_2^2 - \gamma_2 \geq 0.703 \\ x_1 - 2x_2 - q_1^1 \leq 3,\; x_1 - 2x_2 - q_1^2 \leq -7 \\ -2x_1 - x_2 - q_2^1 \leq -7,\; -2x_1 - x_2 - q_2^2 \leq -17 \\ -x_1 + 3x_2 \leq 21, 4x_1 + 3x_2 \leq 45, x_1 + 3x_2 \leq 27, 3x_1 + x_2 \leq 30 \\ x_1, x_2, \gamma_1, \gamma_2 \geq 0 \end{array} \right\} \tag{23}$$

When the model above (23) is solved using the MATLAB-R2023 program, $\gamma_1 = \gamma_2 = 0$ is the outcome. This suggests that the Pareto-optimal solution to IF-MOLDMP (20) is provided by the found solutions $\mathbf{x_1 = 6, x_2 = 7}$. The suggested method, however, allows decision-makers to select the goal that must be accomplished first in order of importance. For instance, as $\delta \nearrow$(increases), the value of $f_1(\mathbf{x}) \nearrow$ and the value of $f_2(\mathbf{x}) \searrow$ (decreases), etc.

8. CONCLUSION

In this paper, the penalized intuitionistic fuzzy goal programming strategy has been proposed for finding Pareto-optimal solutions to the IF-MOLDMP in an intuitionistic fuzzy decision environment. When applied to optimization problems in an imprecise environment, the IFO methodology is among the most effective methods available, yielding more satisfactory outcomes than fuzzy and classical optimizations. The significant contribution of the proposed method is the development of an extended intuitionistic fuzzy interactive technique to determine the most preferred Pareto-optimal for IF-MOLDMP in an intuitionistic fuzzy environment. This technique combines the penalty function method with an appropriate intuitionistic fuzzy aggregation operator. When compared to the existing IFO method, the proposed method can choose from a set of compromise solutions that are both efficient and meet the DM's preference for IF-MOLDMP. Furthermore, the advantages of the proposed method are that there is no need to add an extra zero-one variable, it eliminates the limitation of overestimation or underestimation of the nadir point when establishing a reference point for decision-makers, and it guarantees that the existing solution $\mathbf{x}^*$ satisfies the Pareto-optimally condition.

A future study could focus on applying the proposed solution method to different real-world application problems, such as water resource allocation and inventory control problems, and comparing the outcomes to existing optimization methods.

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Conflict of Interest

On behalf of all authors, the corresponding author state there is no conflict of interest.

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