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ARTICLE

Hilbert Space of Complex-Valued Harmonic Functions in the Unit Disc

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Abstract

The study investigated an extended version of Hilbert space of analytic functions called Hilbert space of complex-valued harmonic functions. It is found that functions in Hilbert space of complex-valued harmonic functions exhibit many properties analogous to its analytic counterpart such as complex-valued harmonic function analogous of norm, equivalent norms, reproducing kernels, growth estimates and Littlewood-Paley Identity Theorem. In particular, the researchers established that several fundamental results known for Hilbert space of analytic functions naturally extend to this broader harmonic framework. Beyond theoretical interest, these findings provide new tools for studying operator theory, potential theory, and approximation processes within the harmonic setting, thereby opening avenues for further research and applications in related areas of Mathematics and applied sciences.

Keywords: Complex-valued harmonic functions; Growth estimates; Hilbert space; Inner product; Integral means; Norm; Reproducing kernel.

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1 Introduction

The study of complex-valued harmonic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} | z| < 1\}$ has a long, tracing back to the seminal work of (Clunie & Sheil-Small, 1984) who first systematically introduced harmonic mappings of the form $f = h + \bar{g}$, where h and g are analytic in $\mathbb{D}$. Such functions naturally generalize analytic mappings, and they have been studied extensively in geometric function theory and operator theory; see, for example, (Dorff & Rolf, 2012; P. Duren, 2004) and the foundational references (Axler et al., 2001; Zhao, 1990).

The Hardy space $H^2(\mathbb{D})$ of analytic functions is a well-established Hilbert space whose power series coefficients belong to $l^2(\mathbb{Z}^+)$; its norm admits several equivalent formulations, and its element enjoys well-known growth and kernel estimates (see Cowen and MacCluer (1995), P. L. Duren (2001), Luery (2013), Romnes (2020), and Shapiro (1993)). Harmonic Hardy spaces $H_h^p(\mathbb{D})$ provide a natural extension where one admits functions of the form f=h+, where h and g are analytic in $\mathbb{D}$. Many classical results for analytic Hardy spaces extend to these spaces (Shapiro, 1993; Zhao, 1990), but the Hilbert case $H_h^2(\mathbb{D})$ is of particular interest because of its inner product structure, reproducing kernel, and operator theoretic applications.

The purpose of this paper is to revisit $H_h^2(\mathbb{D})$, to establish its Hilbert space structure, and to develop self-contained proofs of several of its fundamental properties: equivalent norms, Littlewood-Paley identity, growth estimates, and reproducing kernels. While these results are classical in harmonic analysis, our aim is to highlight their analogues in the harmonic setting in parallel with the analytic case, and to present proofs that are accessible and unified as shown in the preprint (Gebrehana & Geleta, 2024) or https://arxiv.org/abs/2410.22045. The paper is organized as follows. Section 2 defines the space $H_h^2(\mathbb{D})$ and establishes its Hilbert space structure. Section 3 discusses equivalent norms and the Littlewood-Paley identity. Section 4 presents growth estimates and the explicit form of the reproducing kernel. We conclude with remarks on similarities and differences with the analytic case.

2 NORM ON $H_h^2(\mathbb{D})$

In this section, we define a norm on space of complex-valued harmonic functions whose coefficients in the Taylor series representation is square-summable and then show such space is a Hilbert space.

Theorem 2.1 Let f be a complex-valued harmonic function on the unit disc $\mathbb{D}$ given by

Let $f(z) = h(z) + \overline{g(z)}$, where $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are analytic. Suppose

$$H_h^2(\mathbb{D}) = \Big\{ f : \mathbb{D} \to \mathbb{C} : f(z) = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=0}^{\infty} b_n z^n} \quad \text{with } \sum_{n=0}^{\infty} (|a_n|^2 + |b_n|^2) < \infty \Big\}.$$

Then, $\|\cdot\|_{H_h^2(\mathbb{D})} : H_h^2(\mathbb{D}) \to \mathbb{R}$ defined by

$$\|f\|_{H_h^2(\mathbb{D})} = \left(\sum_{n=0}^{\infty} (|a_n|^2 + |b_n|^2) \right)^{1/2}$$

is a norm.

Proof:

(i) Since $\sum_{n=0}^{\infty} (|a_n|^2 + |b_n|^2) \geq 0$, we have $\|f\|^2_{H_h^2(\mathbb{D})} \geq 0$, and

$$\sum_{n=0}^{\infty} (|a_n|^2 + |b_n|^2) = 0 \iff a_n = 0,\, b_n = 0 \iff f = 0.$$

Thus $\|f\|^2_{H_h^2(\mathbb{D})} = 0$ if and only if $f = 0$.

(ii)

$$\begin{aligned} \|\alpha f\|^2_{H_h^2(\mathbb{D})} &= \sum_{n=0}^{\infty} (|\alpha a_n|^2 + |\alpha b_n|^2) \\ &= |\alpha|^2 \sum_{n=0}^{\infty} (|a_n|^2 + |b_n|^2) \\ &= |\alpha|^2 \|f\|^2_{H_h^2(\mathbb{D})}. \end{aligned}$$

Thus $\|\alpha f\|_{H_h^2(\mathbb{D})} = |\alpha| \|f\|_{H_h^2(\mathbb{D})}$.

(iii) Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=0}^{\infty} b_n z^n}$ and $F(z) = \sum_{n=0}^{\infty} A_n z^n + \overline{\sum_{n=0}^{\infty} B_n z^n}$ are in $H_h^2(\mathbb{D})$. Then

$$\begin{aligned} \|f + F\|^2_{H_h^2(\mathbb{D})} &= \sum_{n=0}^{\infty} (|a_n + A_n|^2 + |b_n + B_n|^2) \\ &= \sum_{n=0}^{\infty} \Big[|a_n|^2 + |A_n|^2 + 2\Re(a_n \bar{A}_n) \\ &\quad + |b_n|^2 + |B_n|^2 + 2\Re(b_n \bar{B}_n) \Big]. \end{aligned}$$

So, by Cauchy-Schwartz inequality on the field of complex numbers, we have

$$\begin{aligned} \|f + F\|^2_{H_h^2(\mathbb{D})} &\leq \sum_{n=0}^{\infty} (|a_n|^2 + |b_n|^2) + \sum_{n=0}^{\infty} (|A_n|^2 + |B_n|^2) \\ &\quad + 2\sqrt{\sum_{n=0}^{\infty} |a_n|^2 \sum_{n=0}^{\infty} |A_n|^2} + 2\sqrt{\sum_{n=0}^{\infty} |b_n|^2 \sum_{n=0}^{\infty} |B_n|^2} \\ &\leq \|f\|^2_{H_h^2(\mathbb{D})} + \|F\|^2_{H_h^2(\mathbb{D})} + 2\|f\|_{H_h^2(\mathbb{D})} \|F\|_{H_h^2(\mathbb{D})}. \end{aligned}$$

From which we obtain,

$$\|f + F\|_{H_h^2(\mathbb{D})} \leq \|f\|_{H_h^2(\mathbb{D})} + \|F\|_{H_h^2(\mathbb{D})}.$$

Therefore, from (i), (ii) and (iii), $\|\cdot\|_{H_h^2(\mathbb{D})}$ is a norm on $H_h^2(\mathbb{D})$. $\square$

Remark: Any analytic and conjugate analytic functions are harmonic. Since $h(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic, $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is analytic and $\overline{g(z)} = \overline{\sum_{n=0}^{\infty} b_n z^n}$ is conjugate analytic, we have

$$f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=0}^{\infty} b_n z^n}$$

is harmonic.

Theorem 2.2. Let $H_h^2(\mathbb{D})$ be the space of complex-valued harmonic functions on the unit disc of the form

$$f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=0}^{\infty} b_n z^n},$$

where h and g are analytic in $\mathbb{D}$ and

$$\sum_{n=0}^{\infty} (|a_n|^2 + |b_n|^2) < \infty.$$

Then $H_h^2(\mathbb{D})$ is a Hilbert space with respect to the inner product

$$\langle f, F\rangle = \sum_{n=0}^{\infty}\big(a_n\overline{A}_n + b_n\overline{B}_n\big),$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=0}^{\infty} b_n z^n}$, and $F(z) = \sum_{n=0}^{\infty} A_n z^n + \overline{\sum_{n=0}^{\infty} B_n z^n}$ are in $H_h^2(\mathbb{D})$.

Proof: We define a mapping

$$T : H_h^2(\mathbb{D}) \to \ell^2(\mathbb{Z}^+) \times \ell^2(\mathbb{Z}^+),$$

as $T(f) = (a_n, b_n)_{n\geq 0}$.

It is immediate that T is linear. Moreover,

$$\|T(f)\|^2_{\ell^2\times\ell^2} = \sum_{n=0}^{\infty}|a_n|^2 + \sum_{n=0}^{\infty}|b_n|^2 = \|f\|^2_{H_h^2(\mathbb{D})}.$$

So, T is an isometry. Since every pair of square-summable sequences (a_n), (b_n) defines a function of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=0}^{\infty} b_n z^n},$$

the map T is onto. Hence, T is a linear isometric isomorphism. Since $\ell^2(\mathbb{Z}^+) \times \ell^2(\mathbb{Z}^+)$ is a Hilbert space, its isometric image $H_h^2(\mathbb{D})$ is also a Hilbert space. $\square$

3 EQUIVALENT NORMS

We now consider Complex-valued harmonic functions analogous of equivalent norm by integral means and Littlewood-Paley Identity Theorem which were studied in space of analytic functions.

Theorem 3.1 The norm defined on $H_h^2(\mathbb{D})$(in Theorem 2.1) has another equivalent representation by integral means which is denoted by $M_2^2\,(f, r)$and defined by

$$M_2^2(f, r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^2\, d\theta,$$

where $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=0}^{\infty} b_n z^n}$ on $\mathbb{D}$ and $0 \leq r < 1$.

Proof: Using polar representation of $f = h + \overline{g}$, we get

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta} + \overline{\sum_{n=0}^{\infty} b_n r^n e^{in\theta}}.$$

After some algebraic manipulation, we obtain

$$\begin{aligned}|f(re^{i\theta})|^2 &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n\bar{a}_m r^{n+m} e^{i(n-m)\theta}\\ &+ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_m b_n r^{m+n} e^{i(n+m)\theta}\\ &+ \overline{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_m b_n r^{m+n} e^{i(n+m)\theta}}\\ &+ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_m\bar{b}_n r^{n+m} e^{i(n-m)\theta}.\end{aligned}$$

It is clear that the integral of exponential function $\{e^{i(n-m)\theta}\}_{n=0}^{\infty}$ is 2π when $n = m$ and 0 when $n \neq m$. Multiplying both sides by $\frac{1}{2\pi}$ and integrating with respect to θ from $-\pi$ to π, we get

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^2\, d\theta\\ &= \sum_{n=0}^{\infty}|a_n|^2 r^{2n} + \sum_{n=0}^{\infty}|b_n|^2 r^{2n}\\ &\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} 2\Re\big(a_n b_m r^{n+m} e^{i(n+m)\theta}\big)\, d\theta\\ &= \sum_{n=0}^{\infty}\big(|a_n|^2 + |b_n|^2\big) r^{2n}\\ &\quad + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{2r^{n+m}}{2\pi}\Re\Big(a_n b_m \int_{-\pi}^{\pi} e^{i(n+m)\theta}\, d\theta\Big)\\ &= \sum_{n=0}^{\infty}\big(|a_n|^2 + |b_n|^2\big) r^{2n}.\end{aligned}$$

Therefore,

$$M_2^2(f, r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^2\, d\theta = \sum_{n=0}^{\infty}\big(|a_n|^2 + |b_n|^2\big) r^{2n}.$$

To complete the proof, we need to show $\|f\|_{H_h^2(\mathbb{D})} = \lim_{r\to 1^-} M_2(f, r)$. From the above equation we have

$$M_2^2(f, r) = \sum_{n=0}^{\infty}\big(|a_n|^2+|b_n|^2\big) r^{2n} \leq \sum_{n=0}^{\infty}\big(|a_n|^2+|b_n|^2\big) = \|f\|^2_{H_h^2(\mathbb{D})}$$

whenever $f \in H_h^2(\mathbb{D})$ and $0 \leq r < 1$. So $M_2(f, r)$ is bounded by the $H_h^2(\mathbb{D})$-norm.

It remains to show that whenever $\lim_{r\to 1^-} M_2^2(f, r) = M < \infty$, then the partial sum of the series $M_2^2(f, r) = \sum_{n=0}^{\infty}\big(|a_n|^2 + |b_n|^2\big) r^{2n}$ are bounded on the unit disc by M^2:

$$\sum_{n=0}^{N}\big(|a_n|^2 + |b_n|^2\big) r^{2n} \leq \sum_{n=0}^{\infty}\big(|a_n|^2 + |b_n|^2\big) r^{2n} \leq M^2.$$

As $r \to 1^-$, this partial sum converges to functions in $H_h^2(\mathbb{D})$, which must therefore be bounded by M^2 as well. If every partial sum of Taylor series representation of functions in $H_h^2(\mathbb{D})$ is bounded on the unit disc by M^2, then this is also true for the series. This completes the proof.

Corollary 3.2. The space of bounded complex-valued harmonic functions $f = h + \overline{g}$ on $H_h^\infty(\mathbb{D})$ is a subset of $H_h^2(\mathbb{D})$.

Proof: Let $f \in H_h^\infty(\mathbb{D})$. Then $\|f\|_{H_h^\infty(\mathbb{D})} = \sup_{z\in\mathbb{D}} |f(z)|$. Now,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^2\, d\theta &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sup|f(re^{i\theta})|\right)^2 d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \|f\|^2_{H_h^\infty(\mathbb{D})}\, d\theta \\ &= \|f\|^2_{H_h^\infty(\mathbb{D})},\end{aligned}$$

which holds true for every $0 < r < 1$. So for any $f \in H_h^\infty(\mathbb{D})$ we get

$$\lim_{r\to 1^-} M_2^2(f,r) \le \|f\|^2_{H_h^\infty(\mathbb{D})}.$$

Hence, $f \in H_h^2(\mathbb{D})$. $\square$

The following theorem is Littlewood-Paley identity theorem for space of complex-valued harmonic functions. It provides another expression for the $H_h^2(\mathbb{D})$-norm.

Theorem 3.3. For every complex-valued harmonic function $f = h + \overline{g} \in H_h^2(\mathbb{D})$ we have

$$\begin{aligned}\|f\|^2_{H_h^2(\mathbb{D})} &= |h(0) + \overline{g(0)}|^2 \\ &\quad + 2\int_{\mathbb{D}} |h'(z) + \overline{g'(z)}|^2 \log\frac{1}{|z|}\, dA(z),\end{aligned}$$

where dA denotes the normalized measure on $\mathbb{D}$; i.e., $dA = \frac{1}{\pi}\,dx\,dy = \frac{1}{\pi} r\,dr\,d\theta$.

Proof: We start by considering the right hand side of the equation in Theorem 3.3. Using the polar form of f, we obtain

$$\begin{aligned}&|(h+\overline{g})(0)|^2 + 2\int_{\mathbb{D}} |h'(z) + \overline{g'(z)}|^2 \log\frac{1}{|z|}\, dA \\ &= |h(0) + \overline{g(0)}|^2 \\ &\quad + 2\int_{-\pi}^{\pi} \frac{1}{\pi}\int_0^1 |(h' + \overline{g'})(re^{i\theta})|^2 \left(\log\frac{1}{r}\right) r\,dr\,d\theta.\end{aligned}$$

Interchanging the two integrals (which can be justified by Fubini's theorem), we have

$$\begin{aligned}&|h(0) + \overline{g(0)}|^2 + 2\int_{\mathbb{D}} |h'(z) + \overline{g'(z)}|^2 \log\frac{1}{|z|}\, dA \\ &= |h(0) + \overline{g(0)}|^2 \\ &\quad + 2\int_0^1 \left(\frac{1}{\pi}\int_{-\pi}^{\pi} |(h' + \overline{g'})(re^{i\theta})|^2\, d\theta\right)\left(\log\frac{1}{r}\right) r\,dr.\end{aligned}$$

Applying simple algebraic manipulations, we obtain

$$\begin{aligned}&|h(0) + \overline{g(0)}|^2 + 2\int_{\mathbb{D}} |h'(z) + \overline{g'(z)}|^2 \log\frac{1}{|z|}\, dA \\ &= |h(0) + \overline{g(0)}|^2 \\ &\quad + 4\int_0^1 M_2^2(h'(z) + \overline{g'(z)}, r)\left(\log\frac{1}{r}\right) r\,dr.\end{aligned}$$

Replacing by the Taylor series representation, we get

$$\begin{aligned}&|h(0) + \overline{g(0)}|^2 + 2\int_{\mathbb{D}} |h'(z) + \overline{g'(z)}|^2 \log\frac{1}{|z|}\, dA \\ &= |h(0) + \overline{g(0)}|^2 \\ &\quad + 4\int_0^1 \sum_{n=1}^{\infty}\left(n^2|a_n|^2 + n^2|b_n|^2\right) r^{2n-2}\left(\log\frac{1}{r}\right) r\,dr \\ &= |h(0) + \overline{g(0)}|^2 \\ &\quad + 4\sum_{n=1}^{\infty} n^2\left(|a_n|^2 + |b_n|^2\right)\int_0^1 r^{2(n-1)}\left(\log\frac{1}{r}\right) r\,dr \\ &= |h(0) + \overline{g(0)}|^2 \\ &\quad + 4\sum_{n=1}^{\infty} n^2\left(|a_n|^2 + |b_n|^2\right)\frac{1}{4n^2} \\ &= |h(0) + \overline{g(0)}|^2 + \sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right) \\ &= \|f\|^2_{H_h^2(\mathbb{D})}.\end{aligned}$$

From which we obtain,

$$|h(0) + \overline{g(0)}|^2 + 2\int_{\mathbb{D}} |h'(z) + \overline{g'(z)}|^2 \log\frac{1}{|z|}\, dA = \|f\|^2_{H_h^2(\mathbb{D})}.$$

This completes the proof.

4 GROWTH ESTIMATES AND KERNELS

The analogous growth estimates and reproducing kernels on space of complex-valued harmonic functions in the unit disc can be obtained as follows:-

Theorem 4.1 (Growth estimate). For each $z \in \mathbb{D}$, and $f = h + \overline{g} \in H_h^2(\mathbb{D})$ it holds that $|f(z)| \le \frac{2\|f\|_{H_h^2(\mathbb{D})}}{\sqrt{1-|z|^2}}$.

Proof: By applying the triangle inequality for the modulus and Cauchy-Schwarz inequality to the complex-valued harmonic function f, for each $z \in \mathbb{D}$, we obtain

$$\begin{aligned}|f(z)| &= \left|\sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=0}^{\infty} b_n z^n}\right| \\ &\le \sum_{n=0}^{\infty} |a_n||z|^n + \sum_{n=0}^{\infty} |b_n||z|^n \\ &\le \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} |z|^{2n}\right)^{\frac{1}{2}} \\ &\quad + \left(\sum_{n=0}^{\infty} |b_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} |z|^{2n}\right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{n=0}^{\infty} |a_n|^2\right)^{\frac{1}{2}} + \left(\sum_{n=0}^{\infty} |b_n|^2\right)^{\frac{1}{2}}\right]\left(\sum_{n=0}^{\infty} |z|^{2n}\right)^{\frac{1}{2}} \\ &\le \left(\|h\|_{H_h^2(\mathbb{D})} + \|g\|_{H_h^2(\mathbb{D})}\right)\left(\sum_{n=0}^{\infty} |z|^{2n}\right)^{\frac{1}{2}} \\ &= \frac{\|h\|_{H_h^2(\mathbb{D})} + \|g\|_{H_h^2(\mathbb{D})}}{\sqrt{1-|z|^2}}.\end{aligned}$$

But then, $\|h\|_{H^2_h(\mathbb{D})} \leq \|f\|_{H^2_h(\mathbb{D})}$ and $\|g\|_{H^2_h(\mathbb{D})} \leq \|f\|_{H^2_h(\mathbb{D})}$, which gives

$$|f(z)| \leq \frac{2\|f\|_{H^2_h(\mathbb{D})}}{\sqrt{1-|z|^2}}. \quad \square$$

Theorem 4.2 (Reproducing kernel). Suppose $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc in the complex plane and $H^2_h(\mathbb{D})$ the Hilbert space of complex-valued harmonic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=0}^{\infty} b_n z^n},$$

with

$$\|f\|^2_{H^2_h(\mathbb{D})} = \sum_{n=0}^{\infty} |a_n|^2 + \sum_{n=0}^{\infty} |b_n|^2.$$

Then for each $\alpha \in \mathbb{D}$ the evaluation function $f \longmapsto f(\alpha)$ is bounded and the reproducing kernel is given by

$$K_\alpha(z) = \frac{1}{1-\bar{\alpha}z} + \frac{1}{1-\alpha\bar{z}}, \quad (|\bar{\alpha}z| < 1),$$

satisfying

$$\langle f, K_\alpha \rangle = f(\alpha)$$

for all $f \in H^2_h(\mathbb{D})$; moreover this kernel is unique.

Proof: Define $e_n(z) = z^n$ and $h_n(z) = \bar{z}^n$ for $n \geq 0$. Recall that for

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=0}^{\infty} b_n z^n},$$

$$g(z) = \sum_{n=0}^{\infty} c_n z^n + \overline{\sum_{n=0}^{\infty} d_n z^n},$$

in $H^2_h(\mathbb{D})$ the inner product of f and g is given by,

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \bar{c}_n + \sum_{n=0}^{\infty} b_n \bar{d}_n.$$

This gives $\langle e_n, e_m \rangle = \delta_{nm}$, $\langle h_n, h_m \rangle = \delta_{nm}$ and $\langle e_n, h_m \rangle = 0$ where δ_{nm} represents the Kronecker delta. Thus, $\{e_n\}_{n\geq 0} \cup \{h_n\}_{n\geq 0}$ is an orthonormal basis of $H^2_h(\mathbb{D})$.

Now, by the standard reproducing kernel Hilbert space formula,

$$\begin{aligned} K_\alpha(z) &= \sum_{n=0}^{\infty} e_n(z)\overline{e_n(\alpha)} + \sum_{n=0}^{\infty} h_n(z)\overline{h_n(\alpha)} \\ &= \sum_{n=0}^{\infty} z^n \bar{\alpha}^n + \sum_{n=0}^{\infty} \bar{z}^n \overline{\bar{\alpha}^n} \\ &= \sum_{n=0}^{\infty} \bar{\alpha}^n z^n + \sum_{n=0}^{\infty} \alpha^n \bar{z}^n. \end{aligned}$$

Each series converges for $|\alpha z| < 1$, giving the closed form

$$K_\alpha(z) = \frac{1}{1-\bar{\alpha}z} + \frac{1}{1-\alpha\bar{z}}.$$

To show K_α is in $H^2_h(\mathbb{D})$, the coefficient sequences of K_α are $(\bar{\alpha}^n)_{n\geq 0}$ and $(\alpha^n)_{n\geq 0}$, so

$$\|K_\alpha\|^2_{H^2_h(\mathbb{D})} = \sum_{n=0}^{\infty} |\alpha|^{2n} + \sum_{n=0}^{\infty} |\alpha|^{2n} = \frac{2}{1-|\alpha|^2} < \infty.$$

Hence, K_α is in $H^2_h(\mathbb{D})$ and the evaluation is bounded.

For $f(z) = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=0}^{\infty} b_n z^n} \in H^2_h(\mathbb{D})$ we have

$$\langle f, K_\alpha \rangle = \sum_{n=0}^{\infty} a_n \alpha^n + \sum_{n=0}^{\infty} b_n \bar{\alpha}^n = f(\alpha).$$

So, the reproducing kernel property holds.

Conversely, if for all $\alpha \in \mathbb{D}$ the evaluation $H^2_h(\mathbb{D}) \ni f \longmapsto f(\alpha)$ is a bounded linear functional on $H^2_h(\mathbb{D})$, then by Riesz Representation theorem, there exists a function L_α in $H^2_h(\mathbb{D})$ with the property,

$$f(\alpha) = \langle f, L_\alpha \rangle.$$

If another L_α reproduces evaluation, then $\langle f, K_\alpha - L_\alpha \rangle = 0$ for all $f \in H^2_h(\mathbb{D})$. Taking $f = K_\alpha - L_\alpha$ we get $\|K_\alpha - L_\alpha\|^2_{H^2_h(\mathbb{D})} = 0$. Hence $K_\alpha = L_\alpha$, which shows the kernel is unique.

5 Conclusion

In summary, we have developed a Hilbert space framework for complex-valued harmonic functions on the unit disc, analogous to the well-established analytic setting. An equivalent norm representation was derived in terms of integral means, providing a natural structure for further analysis. Within this framework, we established a harmonic analogue of the Littlewood–Paley Identity Theorem and derived comparable growth estimates for complex-valued harmonic functions, thereby extending classical results from analytic function theory to the harmonic setting.

Furthermore, we introduced the reproducing kernel associated with this Hilbert space, which plays a central role in functional analysis by enabling evaluation functionals and facilitating the study of bounded linear operators. This kernel provides a powerful tool for analyzing the geometry and operator theory of the space.

These results not only enrich the theory of harmonic function spaces but also open new avenues for research. Potential directions include the study of composition operators, multipliers, and dual spaces, as well as applications in potential theory and approximation theory. The harmonic framework developed here may also have implications in related fields such as signal processing and mathematical physics, where harmonic functions naturally arise.

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