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ARTICLE

Alpha Power Transformed Half Normal Distribution with Its Properties and Applications

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Abstract

The study aims to propose a new continuous probability distribution named Alpha Power Transformed Half-Normal Distribution by applying alpha power transformation method to Half normal distribution. Several vigorous statistical and mathematical properties of new distribution are derived. Simulation studies are conducted to evaluate their performance, and comparative model fitting is carried out using real and simulated datasets. The study employs the alpha power transformation technique, simulation experiments, and goodness-of-fit measures to compare the new distribution with five existing distributions. Results indicate that the proposed model provides a superior fit to two real-world datasets and one simulated dataset compared to five competing distributions. The hazard function of the new distribution exhibits highly flexible forms, enhancing its applicability to various lifetime and reliability data. The Alpha Power Transformed Half-Normal Distribution represents a significant contribution to probability and statistical theory. Its flexibility and strong empirical performance suggest its potential for broad application, and further testing on diverse life datasets is recommended.

Keywords: Alpha power transformation; New distribution; Half-normal distribution; Probability and Statistics.

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1 Introduction

Probability distributions perform a decisive role in data-driven decision-making and analysis. Their applications span numerous fields, including physics, computer science, public health, medicine, insurance, reliability analysis, survival analysis, signal processing, communications, and engineering. However, classical distributions often fail to adequately represent diverse real-world datasets, highlighting the need to generalize existing distributions (Mohiuddin & Kannan, 2022). The generalization of existing probability models has become an important area of research in statistical theory (Mohiuddin & Kannan, 2021). For example, Yousof H.M. (2019) presented a new lifetime and regression model with real-world applications, while Alizadeh et al. (2021) proposed a new two-parameter lifetime model and derived its statistical properties. Similarly, Altun et al. (2018) introduced a generalized lifetime model derived from the half-normal distribution, including its regression form and properties.

The Half-Normal (HN) distribution is a special case of the normal distribution restricted to non-negative values (Nadarajah & Kotz, 2006). It has been widely used in lifetime data analysis, quality control, and various other disciplines (Wallner, 2020). Despite its usefulness, the HN distribution is limited by its rigidity, with its shape is controlled solely by a single scale parameter, making it incapable of modeling skewed, heavy-tailed, or otherwise complex datasets. Moreover, its hazard rate is monotonic and non-decreasing, which restricts its applicability in modeling systems that exhibit non-monotonic hazard behaviors, such as bathtub-shaped or decreasing hazard rates commonly seen in reliability and survival studies.

To address these limitations, numerous extensions of the HN distribution had been proposed. For instance, the Square-Transformed Half-Normal distribution, derived by (Bousseba & Sakr, 2025), has a fixed shape because squaring does not introduce any additional parameter. Its tail behavior, skewness, and kurtosis are still inflexible and unable to adjust to actual data that calls for flexibility and various hazard shapes. The Alpha Power Transformed Half-Normal distribution is motivated by this constraint and overcomes it by adding a shape parameter. Similarly, the Exponentiated Half-Normal Distribution (EHND) introduced by Cordeiro et al. (2016) adds a shape parameter to enhance flexibility in modeling various data structures. Likewise, the Kumaraswamy Half-Normal Distribution (KHND) developed by Alizadeh et al. (2020) incorporates the Kumaraswamy distribution as a generator, providing additional adaptability for diverse data patterns. Although these generalizations improve the model's flexibility, they still may not adequately capture certain complex datasets or hazard rate shapes, such as bathtub or decreasing hazard functions frequently observed in reliability studies.

Several other one-parameter lifetime distributions have been developed. For example, the XGamma distribution proposed by Sen et al. (2016) combines exponential and gamma distributions, while the XLindley distribution introduced by Chouia and Zeghdoudi (2021) mixes exponential and Lindley distributions. Both models exhibit desirable statistical properties, but are limited by their single-parameter structure, which restricts their ability to model a wide range of data behaviors. In an effort to enhance flexibility, Alshenawy (2022) developed the three-parameter X-Gamma Inverse Weibull distribution, whose density is a linear combination of Inverse Weibull densities. Although this model can capture increasing, decreasing, and unimodal hazard rates, it still fails to represent bathtub-shaped hazards adequately. Therefore, there remains a need for a more flexible two-parameter model capable of fitting datasets with various hazard rate forms.

To fill this gap, we introduce a new distribution with additional shape parameter from the half-normal distribution using the alpha power transformation (APT) method. The alpha power transformation technique, originally developed by Mahdavi and Kundu (2017), provides a systematic way to generate flexible probability distributions. Later, (Alizadeh et al., 2021) extended this concept to define a broader class of modified distributions.

The primary objective of this study is to propose a new, lithe two-parameter model called the Alpha Power Transformed Half-Normal Distribution (APTHND). We derive and prove its statistical properties, and demonstrate its performance through data fitting, simulation studies, and graphical analyses using R software version 4.3.1 (R Core Team, 2023).

2 The Base Half Normal Distribution

The half-normal distribution is a probability distribution that is a variant of the normal distribution, but only extends in one direction from the mean. It is a widely used distribution in applied statistics (Wallner, 2020). It provides a useful framework for modeling and analyzing non-negative continuous data that exhibits one-sided patterns.

Let the random variable represented by X follow the half normal distribution with mean zero and variance σ^2. Then, the corresponding probability density and cumulative distribution functions are given as (Bader et al., 2022).

$$f(x) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-x^2/(2\sigma^2)}, \quad x > 0, \tag{1}$$

$$F(x) = \operatorname{erf}\left(\frac{x}{\sigma\sqrt{2}}\right), \quad x > 0, \tag{2}$$

where $\operatorname{erf}(\cdot)$ is the error function.

In this paper, we introduce a new distribution by transforming the half normal distribution using the alpha power transformation technique proposed by Mahdavi and Kundu (2017).

3 Alpha Power Transformed Half-Normal Distribution

The Alpha Power Transformation (APT) approach was introduced by Mahdavi and Kundu (2017) to increase flexibility of the new class of distributions. Given base distribution with $f(x)$ and $F(x)$, the APT transformation generally gives new probability density function $f_{APT}(x)$ and cumulative distribution function $F_{APT}(x)$:

$$f_{\text{APT}}(x) = \begin{cases} \log(\alpha) f(x) [\alpha F(x)]^{\alpha - 1}, & \text{if } \alpha > 0, \alpha \neq 1, \\ f(x), & \text{if } \alpha = 1, \end{cases} \tag{3}$$

$$F_{\text{APT}}(x) = \begin{cases} \dfrac{\alpha F(x) - 1}{\alpha - 1}, & \text{if } \alpha > 0, \alpha \neq 1, \\ F(x), & \text{if } \alpha = 1. \end{cases} \tag{4}$$

Using APT method given in equations (3 - 4), we introduce the novel distribution which is called Alpha Power Transformed Half Normal (APTHN) distribution from the base half normal distribution given in equations (1- 2).

Theorem 1: A random variable X having an alpha power transformed half normal distribution with two parameters if its CDF and PDF were given as below:

a. The CDF is

$$F_{\text{APTHN}}(x;\alpha,\sigma^2)=\begin{cases}\dfrac{\alpha\,\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)-1}{\alpha-1}, & \text{if } x>0,\ \alpha>0,\ \alpha\neq 1,\\ \text{erf}\left(\dfrac{x}{\sigma\sqrt{2}}\right), & \text{if } x>0,\ \alpha=1.\end{cases} \tag{5}$$

b. The PDF is

$$f_{\text{APTHN}}(x;\alpha,\sigma^2)=\begin{cases}\dfrac{2\alpha\,\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\ln(\alpha)e^{-x^2/(2\sigma^2)}}{\sqrt{\pi}\,\sigma(\alpha-1)}, & \text{if } x>0,\ \alpha>0,\\ & \alpha\neq 1,\\ \dfrac{2}{\sigma\sqrt{\pi}}e^{-x^2/(2\sigma^2)}, & \text{if } x>0,\ \alpha=1.\end{cases} \tag{6}$$

The function $f_{\text{APTHN}}(x;\alpha,\sigma^2)$ is non-negative and has integration equal to one.

Proof: a. If X is a random variable following APTHN distribution, then its CDF is derived by substituting equation (2) in equation (4):

$$\begin{aligned}F_{\text{APTHN}}(x;\alpha,\sigma^2)&=\frac{\alpha^{F(x)}-1}{\alpha-1}\\&=\frac{\alpha^{\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}-1}{\alpha-1}.\end{aligned}$$

b. If X is a random variable following APTHN distribution, then its PDF is obtained by derivation of equation (5):

$$\begin{aligned}f_{\text{APTHN}}(x;\alpha,\sigma^2)&=\frac{d}{dx}\left(F_{\text{APT}}(x;\alpha,\sigma^2)\right)\\&=\frac{d}{dx}\left(\frac{\alpha^{\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}-1}{\alpha-1}\right)\\&=\frac{1}{\alpha-1}\frac{d}{dx}\left(\alpha^{\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}\right)\\&=\frac{1}{\alpha-1}\ln(\alpha)\alpha^{\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}\frac{d}{dx}\left[\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]\\&=\frac{1}{\alpha-1}\ln(\alpha)\alpha^{\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}\cdot\frac{2}{\sqrt{\pi}}e^{-x^2/(2\sigma^2)}\cdot\frac{1}{\sigma\sqrt{2}}\\&=\frac{\sqrt{2}\ln(\alpha)\alpha^{\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}e^{-x^2/(2\sigma^2)}}{\sqrt{\pi}(\alpha-1)\sigma}.\end{aligned}$$

To show integration of PDF is unity:

$$\begin{aligned}&\int_0^\infty f_{\text{APTHN}}(x)\,dx\\&=\int_0^\infty\frac{\sqrt{2}\ln(\alpha)e^{-x^2/(2\sigma^2)}\alpha^{\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}}{\sqrt{\pi}(\alpha-1)\sigma}\,dx\\&=\frac{\sqrt{2}\ln(\alpha)}{\sqrt{\pi}(\alpha-1)\sigma}\int_0^\infty e^{-x^2/(2\sigma^2)}\alpha^{\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}\,dx.\end{aligned}$$

Let $u=\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)$. Then

$$\frac{du}{dx}=\frac{2}{\sqrt{\pi}}e^{-x^2/(2\sigma^2)}\cdot\frac{1}{\sigma\sqrt{2}},\quad dx=\frac{\sqrt{\pi}\sigma}{\sqrt{2}}e^{x^2/(2\sigma^2)}\,du.$$

Thus,

$$\begin{aligned}&\int_0^\infty f_{\text{APTHN}}(x)\,dx\\&=\frac{\sqrt{2}\ln(\alpha)}{\sqrt{\pi}(\alpha-1)\sigma}\int_0^1 e^{-x^2/(2\sigma^2)}\alpha^u\frac{\sqrt{\pi}\sigma}{\sqrt{2}}e^{x^2/(2\sigma^2)}\,du\\&=\frac{\ln(\alpha)}{\alpha-1}\int_0^1\alpha^u\,du\\&=\frac{\ln(\alpha)}{\alpha-1}\left.\frac{\alpha^u}{\ln(\alpha)}\right|_0^1\\&=\frac{1}{\alpha-1}(\alpha^1-\alpha^0)=1.\end{aligned}$$

Theorem 2: Assume a random variable X follows the APTHN distribution, then

The survival function of X is

$$S_{\text{APTHN}}(x;\alpha,\sigma^2)=\begin{cases}\dfrac{\alpha-\alpha^{\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}}{\alpha-1}, & \text{if } x>0,\ \alpha>0,\ \alpha\neq 1,\\ 1-\text{erf}\left(\dfrac{x}{\sigma\sqrt{2}}\right), & \text{if } x>0,\ \alpha=1.\end{cases} \tag{7}$$

The hazard rate function of X is

$$h_{\text{APTHN}}(x;\alpha,\sigma^2)=\begin{cases}\dfrac{2\alpha^{\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}\ln(\alpha)e^{-x^2/(2\sigma^2)}}{\sqrt{\pi}\,\sigma\left(\alpha-\alpha^{\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}\right)}, & \text{if } x>0,\ \alpha>0,\\ & \alpha\neq 1,\\ \dfrac{2e^{-x^2/(2\sigma^2)}}{\sqrt{\pi}\,\sigma\left[1-\text{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]}, & \text{if } x>0,\ \alpha=1.\end{cases} \tag{8}$$

Proof: For the random variable X with $\alpha>0,\alpha\neq 1$, its survival function is:

$$\begin{aligned}S_{\text{APTHN}}(x;\alpha,\sigma^2)&=\int_x^\infty f_{\text{APTHN}}(t;\alpha,\sigma^2)\,dt\\&=\int_x^\infty\frac{\sqrt{2}\ln(\alpha)e^{-t^2/(2\sigma^2)}\alpha^{\text{erf}\left(\frac{t}{\sigma\sqrt{2}}\right)}}{\sqrt{\pi}(\alpha-1)\sigma}\,dt\\&=\frac{\sqrt{2}\ln(\alpha)}{\sqrt{\pi}(\alpha-1)\sigma}\int_x^\infty e^{-t^2/(2\sigma^2)}\alpha^{\text{erf}\left(\frac{t}{\sigma\sqrt{2}}\right)}\,dt.\end{aligned}$$

Let $u = \operatorname{erf}\left(\frac{t}{\sigma\sqrt{2}}\right)$. Then

$$\frac{du}{dt} = \frac{2}{\sqrt{\pi}} e^{-t^2/(2\sigma^2)} \cdot \frac{1}{\sigma\sqrt{2}}, \quad dt = \frac{\sqrt{\pi}\sigma}{\sqrt{2}} e^{t^2/(2\sigma^2)}\, du.$$

Thus,

$$\begin{aligned} S_{\text{APTHN}}(x;\alpha,\sigma^2) &= \frac{\sqrt{2}\ln(\alpha)}{\sqrt{\pi}(\alpha-1)\sigma} \int_{u(x)}^{1} e^{-t^2/(2\sigma^2)} \alpha^u \frac{\sqrt{\pi}\sigma}{\sqrt{2}} e^{t^2/(2\sigma^2)}\, du \\ &= \frac{\ln(\alpha)}{\alpha-1} \int_{\operatorname{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}^{1} \alpha^u\, du \\ &= \frac{\ln(\alpha)}{\alpha-1} \left. \frac{\alpha^u}{\ln(\alpha)} \right|_{\operatorname{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}^{1} \\ &= \frac{1}{\alpha-1}\left(\alpha^1 - \alpha^{\operatorname{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}\right) \\ &= \frac{\alpha - \alpha^{\operatorname{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}}{\alpha-1}. \end{aligned}$$

The hazard function of random variable X is:

$$h_{\text{APTHN}}(x;\alpha,\sigma^2) = \begin{cases} \dfrac{2\alpha^{\operatorname{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)} \ln(\alpha) e^{-x^2/(2\sigma^2)}}{\sqrt{\pi}\,\sigma\left(\alpha - \alpha^{\operatorname{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}\right)}, & \text{if } x > 0,\ \alpha > 0, \\ & \alpha \neq 1, \\ \dfrac{2e^{-x^2/(2\sigma^2)}}{\sqrt{\pi}\,\sigma\left[1 - \operatorname{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]}, & \text{if } x > 0,\ \alpha = 1. \end{cases} \tag{9}$$

4 Plots of Alpha Power Transformed Half Normal Distribution

The plots of the probability density function, cumulative distribution function, survival function, and hazard function of the APTHN distribution are shown in three cases. These are when variance is equal to one, when variance is greater than one, and when variance is less than one.

Case 1. Case of variance of half normal distribution $\sigma^2 = 1$ with $0 < \alpha \geq 1$ The plots in Figure 1 show that as alpha value gets very small ($\alpha < 1$) while variance is kept at unity, the curve of pdf loses its bell shape until it looks like the decaying exponential density function. The mode grows. The respective cdf quickly grows and converges to one as alpha gets smaller as compared to the base half normal case (alpha = 1). The black colored plots refer to the base half normal distribution.

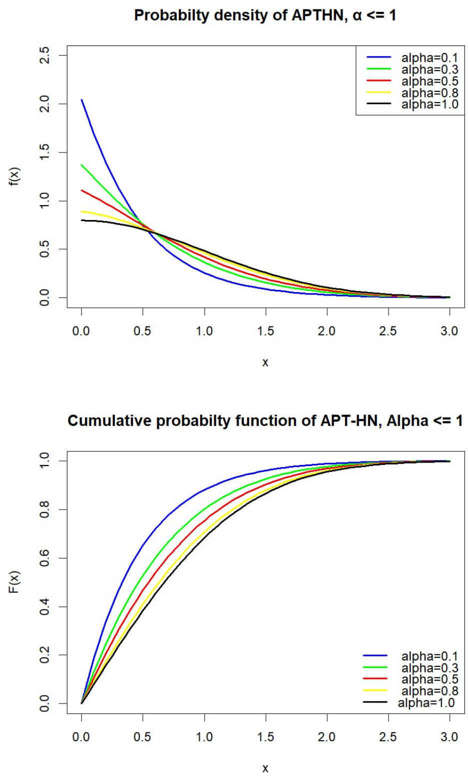


Figure 1: Plots of density and cumulative distribution functions of APTHN for $\alpha \leq 1, \sigma^2 = 1$.

Plots in Figure 2 show that as alpha value gets larger (alpha > 1) while variance is fixed to be one, the modal point moves to the right, generates high left skewness and low right skewness. The cdf grows and converges to 1 with lower growth rate as compared to the base distribution.

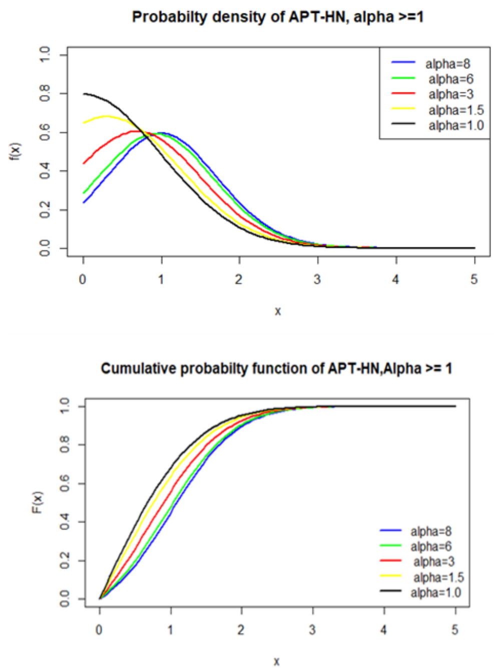


Figure 2: Plots of density and cumulative distribution functions of APTHN for $\alpha \geq 1, \sigma^2 = 1$.

Figures 3-4 illustrates the respective survival and hazard functions. The survival function is generally decreasing and converges to limit zero. Its rate of decay increases with increasing alpha value. The hazard function has bump shapes for alpha smaller than one and is ever increasing for alpha greater than one. These are new interesting features of the hazard function we can expect from the new APTHN distribution. The black-colored plots refer to the base half-normal distribution.

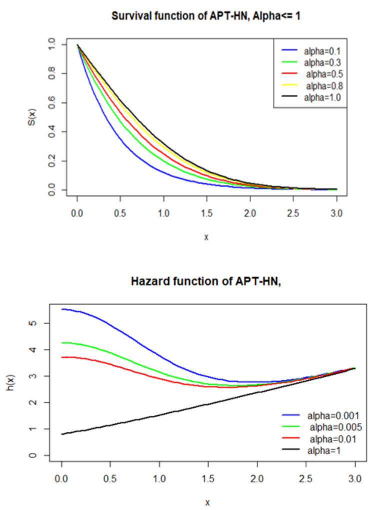


Figure 3: Plots of survival and hazard function of APTHN for $0 < \alpha \leq 1, \sigma^2 = 1$

Case 2. Case of variance of half normal distribution $\sigma^2 > 1$ with $0 < \alpha \geq 1$ Figures 5-8 show that the plots of the PDF, CDF, survival and hazard functions. Similar curves and properties are illustrated by the new distribution except over the larger domain of X. This is expected since larger variance is taken. The black colored plots refer to the base half normal distribution.

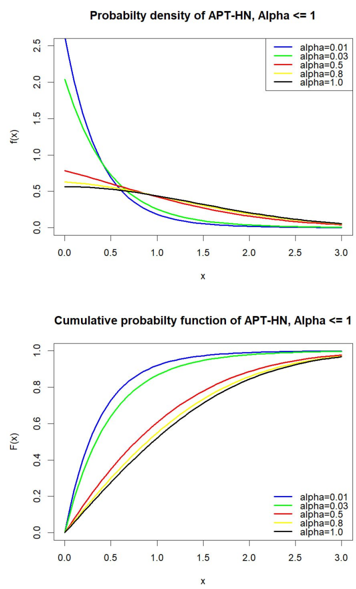


Figure 5: Plots of survival and hazard function of APTHN for $0 < \alpha \leq 1, \sigma^2 = 1$

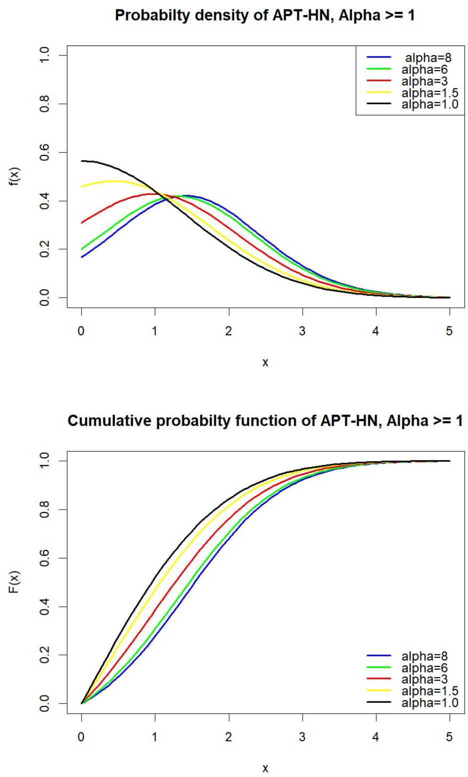


Figure 6: Plots of density and cumulative distribution functions for $\alpha \geq 1, \sigma^2 = 2$.

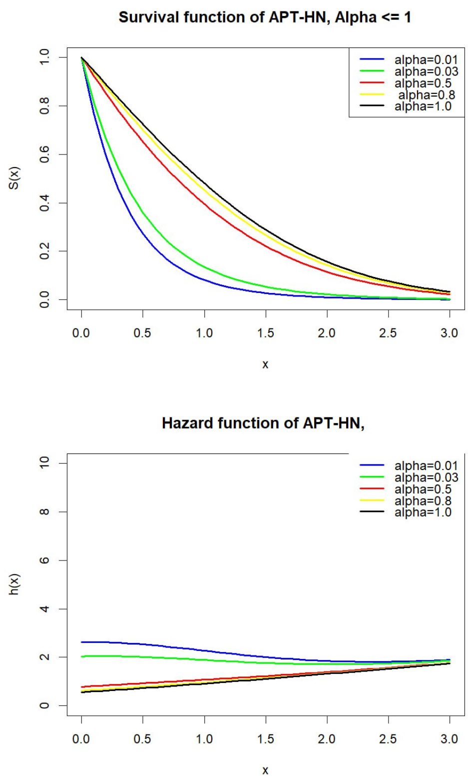


Figure 7: Plots of survival and hazard functions for $0 < \alpha \leq 1$, $\sigma^2 = 2$

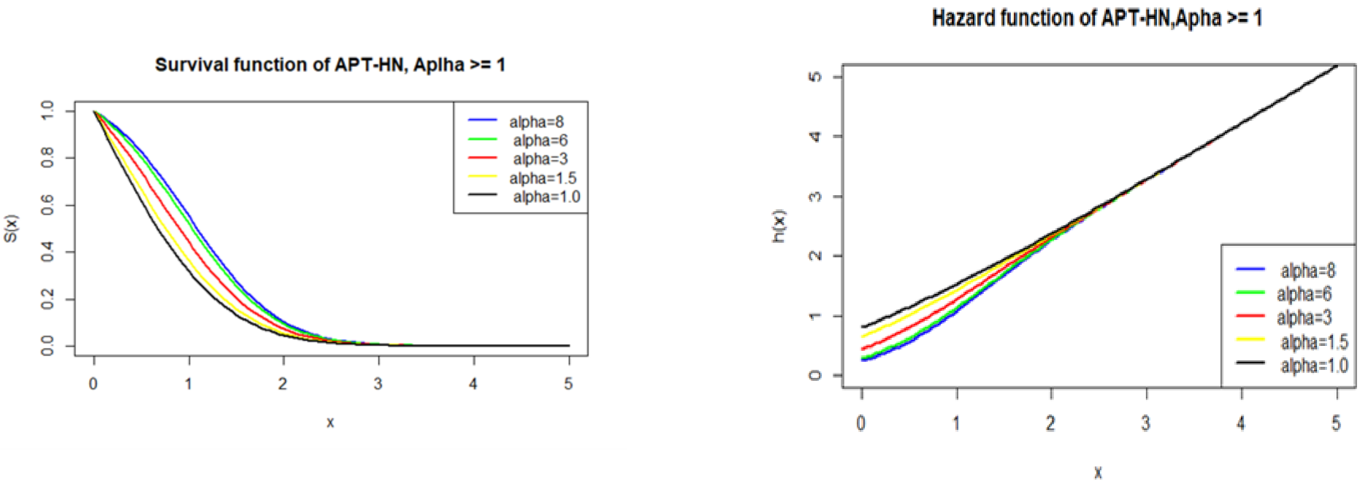


Figure 4: Plots of survival and hazard function of APTHN for $\alpha \geq 1, \sigma^2 = 1$.

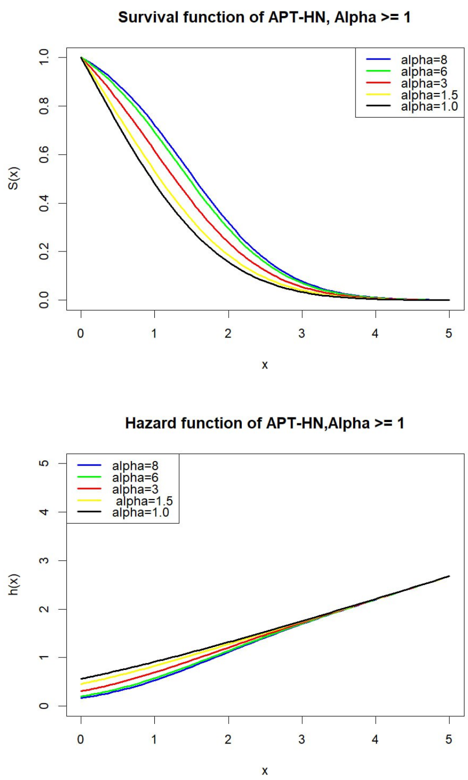


Figure 8: Plots of survival and hazard functions for $\alpha \geq 1$, $\sigma^2 = 2$.

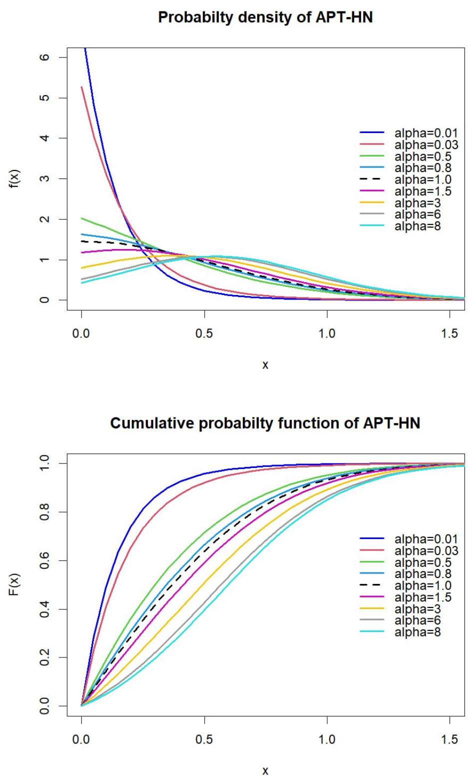
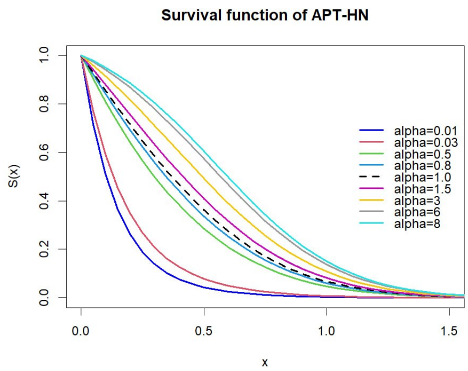


Figure 9: Plots of density and cumulative distribution functions for $0 < \alpha \geq 1, \sigma^2 = 0.3$.

Case 3. Case of variance of half normal distribution $\sigma^2 < 1$ with $0 < \alpha \geq 1$ The plots in Figure 9-10 demonstrate that the curves of the probability density, cumulative distribution, survival and hazard functions of the new distribution shrink to smaller domain of X and re-shaping the skewness of the distribution. This is expected for such smaller variance values less than one. The hazard function shows flexible curves that can be used for modelling life data having such behavior including bumping shapes. Black colored plots refer to base distribution.



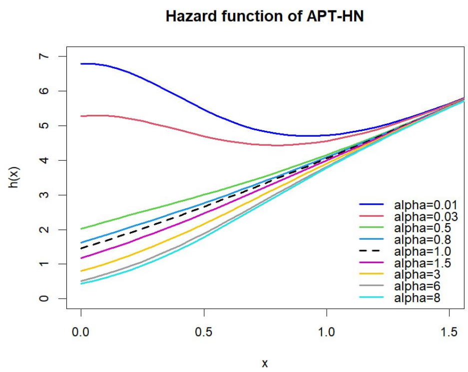


Figure 10: Plots of survival and hazard functions for $0 < \alpha \geq 1$, $\sigma^2 = 0.3$.

The plots 1-10 demonstrate the shapes of the probability density, cumulative distribution, survival, and hazard functions of the new distribution. Various interesting features are shown by varying the two parameters. One feature to emphasize is the plot with parameters = 0.01 and = 0.3 with blue color in Figure 9-10. The plot shows a decaying nature in the pdf, the fastest growth in the cdf, a fast decline in survival probability, and the largest hazard rate with a bumping shape. The hazard rate has inflection points.

In data fitting practices, for example, for right-skewed data, we often require such a new distribution as an alternative model to the exponential distribution whose hazard rate is constant. Thus, we believe that the new probability distribution (APTHND) is different from the base distribution and is so flexible, showing its novelty. We derive detailed properties of the model APTHN distribution in next section.

5 Properties of Alpha Power Transformed Half Normal Distribution

5.1 Quantile Function and Median

Assume X is the random variable that follows the APTHN distribution with location zero, variance sigma square and shape alpha, then the quantile function Q(u) of X is given by:

$$Q\left(u\right) = F^{-1}\left(\frac{\alpha^{erf\left(u/\sigma\sqrt{2}\right)} - 1}{\alpha - 1}\right) \tag{10}$$

If u is uniform distribution with interval (0,1), then X APTHN, the pth quantile function of APTHN distribution is given by:

$$X_p = \frac{\sqrt{2}\,\sigma\, erf(ln(p\,(\alpha-1)+1))}{ln(\alpha)} \tag{11}$$

Median can be derived as

$$X_{0.5} = \frac{\sqrt{2}\,\sigma\, erf^{-1}(ln(0.5\,(\alpha-1)+1))}{ln(\alpha)} \tag{12}$$

5.2 Moments

Assume there is a random variable X, for r a positive integer and if $X \in L^r$, the r^{th} moment of X is

$$E\left[X^r\right] = \int_{-\infty}^{\infty} x^r f(x)\, dx.$$

Where L^r is the space consisting of all random variables whose r^{th} absolute power is integrable, and for $1 \leq r < \infty$, L^r denotes the set of random variables X such that $E\left[|X|^r\right] < \infty$.

Theorem 3: If a random variable X is from the APTHN distribution, then its rth moment is given by

$$\mu_r' = \frac{\sigma^r \ln\left(\alpha\right)}{\alpha - 1}\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\left(\ln\left(\alpha\right)\right)^k}{k!} I_{r,k} \tag{13}$$

where

$$I_{r,k} = \int_0^{\infty} u^r e^{-u^2 2}\left(\text{erf}\left(u\sqrt{2}\right)\right)^k du \tag{14}$$

Proof If the random variable X had PDF (6), then the rth moments of X, where X follows the APTHN distribution are calculated as

$$\mu_{r=}' \int_0^{\infty} x^r f_{APT}\left(x; \alpha, \sigma^2\right) dx$$

$$\mu_{r=}' \int_0^{\infty} x^r \frac{\sqrt{2}\left(\alpha^{\text{erf}\left(x\sigma\sqrt{2}\right)} * \ln\left(\alpha\right) * e^{-x^2 2\sigma^2}\right)}{\sqrt{\pi}\left(\alpha - 1\right)\sigma} dx$$

$$\mu_{r=}' \frac{\sqrt{2}\ln(\alpha)}{\sqrt{\pi}\left(\alpha-1\right)\sigma}\int_0^{\infty} x^r \alpha^{\text{erf}\left(x\sigma\sqrt{2}\right)} e^{-x^2 2\sigma^2} dx$$

By using the exponential series representation

$\alpha^w = \sum_{k=0}^{\infty}\frac{(\ln(\alpha))^k}{k!} w^k$, we have:

$$\alpha^{\text{erf}\left(x\sigma\sqrt{2}\right)} = \sum_{k=0}^{\infty}\frac{(\ln(\alpha))^k}{k!}\left(\text{erf}\left(x\sigma\sqrt{2}\right)\right)^k \tag{15}$$

$$\mu'_{r=} \frac{\sqrt{2}\ln(\alpha)}{\sqrt{\pi}\,(\alpha-1)\,\sigma}\sum_{k=0}^{\infty}\frac{(\ln(\alpha))^k}{k!}\int_0^{\infty} x^r e^{-x^2 2\sigma^2}\left(\text{erf}\left(x\sigma\sqrt{2}\right)\right)^k dx$$

By setting $u = x/\sigma$, then $\frac{du}{dx} = \frac{1}{\sigma}$ and $\sigma du = dx$, then $x = u\sigma$

$$\mu'_{r=} \frac{\sigma^r\sqrt{2}\ln(\alpha)}{\sqrt{\pi}\,(\alpha-1)}\sum_{k=0}^{\infty}\frac{(\ln(\alpha))^k}{k!}\int_0^{\infty} u^r e^{-u^2 2}\left(\text{erf}\left(u\sqrt{2}\right)\right)^k du$$

$$\mu'_{r=} \frac{\sigma^r\sqrt{2}\ln(\alpha)}{\sqrt{\pi}\,(\alpha-1)}\sum_{k=0}^{\infty}\frac{(\ln(\alpha))^k}{k!}I_{r,k}$$

Where $I_{r,k} = \int_0^{\infty} u^r e^{-u^2 2}\left(\text{erf}\left(u\sqrt{2}\right)\right)^k du$

5.3 Mean and Variance of APTHN Distribution.

For a random variable $X \in L^1$, the mean of X is $E[X] = \int_{-\infty}^{\infty} xf(x)dx$, Expectation of X. This can be easily obtained by putting $r = 1$ from rth moment. It is given as

$$\mu = \mu'_{1=} \frac{\sigma^1\sqrt{2}\ln(\alpha)}{\sqrt{\pi}\,(\alpha-1)}\sum_{k=0}^{\infty}\frac{(\ln(\alpha))^k}{k!}I_{1,k} \quad (16)$$

Similarly; suppose that the random variable $X \in L^2$, then the variance X is the second central moments.

$$Var\ (X) = E\left[(X - E[X])^2\right] = E\left[X^2\right] - E[X]^2$$

Therefore, the variance of random variable X, where X follows the APTHN distribution is given by $Var\ (X) = E\left[X^2\right] - E[X]^2$, Hence $E\left[X^2\right]$ is the second moment (μ'_2) which is easily obtained by putting $r = 2$ from rth moment. Thus it is given by:

$$Var\ (X) = \mu'_2 - (\mu'_1)^2 = \frac{\sigma^2\sqrt{2}\ln(\alpha)}{\sqrt{\pi}\,(\alpha-1)}\sum_{k=0}^{\infty}\frac{(\ln(\alpha))^k}{k!}I_{2,k} - \mu^2 \quad (17)$$

Moment Generating Function (MGF): The MGF of the random variable X is the function $\psi_x(t) = E\left[e^{tx}\right] = \int_{-\infty}^{\infty} e^{tx}f(x)dx$, provided that the expectation occurs for all t in some neighborhood of the origin [9].

The Moment generating Function of random variable X where X follows APTHN distribution is:

$$\psi_x(t) = E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} f_{APT}\left(x;\alpha,\sigma^2\right)dx$$

$$\psi_x(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty} x^r f_{APT}\left(x;\alpha,\sigma^2\right)dx = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu'_r$$

$$\psi_x(t) = \frac{\sigma^r\sqrt{2}\ln(\alpha)}{\sqrt{\pi}\,(\alpha-1)}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{(\ln(\alpha))^k}{k!}I_{r,k}$$

5.4 Characteristic Function

The characteristic function of a random variable X is the function $\varphi_x : \mathbb{R} \to \mathbb{C}$ expressed by: $\varphi_x(t) = E\left[e^{itx}\right] = \int_{-\infty}^{\infty} e^{itx}f(x)dx$, where $\mathbb{R}$ is set of real number and $\mathbb{C}$ complex number (Karr, 1993). The Characteristic function of random variable X, where X follows APTHN distribution is expressed by

$$\varphi_x(t) = E\left(e^{itx}\right) = \int_0^{\infty} e^{itx} f_{APT}\left(x;\alpha,\sigma^2\right)dx$$

$$\varphi_x(t) = \sum_{r=0}^{\infty}\frac{(it)^r}{r!}\int_0^{\infty} x^r f_{APT}\left(x;\alpha,\sigma^2\right)dx = \sum_{r=0}^{\infty}\frac{(it)^r}{r!}\mu'_r$$

Hence, characteristic function becomes:

$$\varphi_x(t) = \frac{\sigma^r\sqrt{2}\ln(\alpha)}{\sqrt{\pi}\,(\alpha-1)}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\frac{(\ln(\alpha))^k}{k!}I_{r,k} \quad (18)$$

Where $i = \sqrt{-1}$ is a complex number

5.5 Order Statistics

Assume $X_1, X_2, \ldots, X_n$ be a random sample taken from APTHN distribution, and let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ denote the order statistics. Then, the probability density function of the random variable $X_{(i)}$ is expressed by:

$$g_{X_i}(x) = \frac{n!}{(i-1)!\,(n-i)!}f(x) * F(x)^{i-1}(1-F(x))^{n-i}$$

The PDF of the random variable $X_{(i)}$ can be expressed as

$$g_{X_i}(x) = \frac{n!}{(i-1)!\,(n-i)!}\frac{\sqrt{2}\left(\alpha^{\text{erf}\left(x\sigma\sqrt{2}\right)} * \ln(\alpha) * e^{-x^2 2\sigma^2}\right)}{\sqrt{\pi}\,(\alpha-1)\,\sigma}\left(\frac{\alpha^{\text{erf}\left(x\sigma\sqrt{2}\right)}-1}{\alpha-1}\right)^{i-1}\left(1-\frac{\alpha^{\text{erf}\left(x\sigma\sqrt{2}\right)}+1}{\alpha-1}\right)^{n-1} \quad (17)$$

In particular, the PDF of the first and last order statistics of the random variable X can be derived as

$$g_{X_1}(x) = \frac{n\sqrt{2}\ln(\alpha)\alpha^{\operatorname{erf}\left(x\sigma\sqrt{2}\right)}e^{-x^2 2\sigma^2}}{\sqrt{\pi}\ (\alpha-1)^n\ \sigma}\ \left(\alpha - \alpha^{\operatorname{erf}\left(x\sigma\sqrt{2}\right)}\right)^{n-1}$$
$$g_{X_n}(x) = \frac{n\sqrt{2}\ln(\alpha)\alpha^{\operatorname{erf}\left(x\sigma\sqrt{2}\right)}e^{-x^2 2\sigma^2}}{\sqrt{\pi}\ (\alpha-1)^n\ \sigma}\ \left(\alpha^{\operatorname{erf}\left(x\sigma\sqrt{2}\right)} - 1\right)^{n-1}$$

6 Maximum Likelihood Estimation of Parameters

The population parameters of the APTHN distribution can be estimated using maximum likelihood estimation (MLE) method. It derivation is shown below.

Suppose $X_1, \ldots, X_n$ be n independently identically distributed random observations from $f_{\text{APTHN}}(\boldsymbol{x};\boldsymbol{\theta})$ with $\boldsymbol{\theta} = (\alpha, \sigma^2) \in \Omega = [0,\infty) \times [0,\infty)$ in parameter space a subset of 2-dimensional $\mathbb{R}^2$. The likelihood function $L(\boldsymbol{\theta}|\boldsymbol{x})$ is given by $L(\boldsymbol{\theta}|\boldsymbol{x}) = \prod_{i=1}^{n} f(x_i;\boldsymbol{\theta})$. The values of $\boldsymbol{\theta}$ that maximizes $L(\boldsymbol{\theta}|\boldsymbol{x})$ is a maximum likelihood estimators (MLEs) $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ [17].

Thus, unique solution MLE $\hat{\boldsymbol{\theta}}$ is obtained by optimization problem approach as:

$$\hat{\boldsymbol{\theta}} = \frac{\arg\max}{\boldsymbol{\theta} \in \Omega} L(\boldsymbol{\theta}|\boldsymbol{x}) \tag{19}$$
$$(\hat{\alpha}, \hat{\sigma}^2) = \frac{\arg\max}{(\alpha,\sigma^2) \in \Omega} \prod_{i=1}^{n} f_{\text{APT}}(\boldsymbol{x};\alpha,\sigma^2) \tag{20}$$

Alternatively, we can find the solution by log-likelihood function. By substituting equation (6) in to equation (21) and taking the logarithm of the function, we found the log-likelihood function $\log(L)$ which is given by:

$$\log(L) = \log\left(\prod_{i=1}^{n} \frac{\sqrt{2}\left(\alpha^{\operatorname{erf}\left(x\sigma\sqrt{2}\right)} * \ln(\alpha) * \exp(-x^2 2\sigma^2)\right)}{\sqrt{\pi}(\alpha-1)\sigma}\right)$$
$$\log(L) = \frac{n}{2} * \log\left(\frac{2}{\pi}\right) + \sum_{i=1}^{n} \operatorname{erf}\left(\frac{x_i}{\sigma\sqrt{2}}\right) * \log(\alpha) + n * \log(\log(\alpha)) - \sum_{i=1}^{n} \frac{x_i^2}{2\sigma^2} - n * \log(\alpha-1) - n * \log(\sigma)$$

By taking partial derivatives for $\log L$ with respect to both α and σ respectively, we get

$$\frac{\partial \log(L)}{\partial \alpha} = \frac{\sum_{i=1}^{n} \operatorname{erf}\left(\frac{x_i}{\sigma\sqrt{2}}\right)}{\alpha} + \frac{n}{\alpha * \ln(\alpha)} - \frac{n}{\alpha - 1}$$
$$\frac{\partial \log(L)}{\partial \sigma} = -\frac{\sqrt{2} * \ln(\alpha) \sum_{i=1}^{n} x_i \exp(-x_i^2 2\sigma^2)}{\sqrt{\pi}\sigma^2} + \sum_{i=1}^{n} \frac{x_i^2}{2\sigma^3} - \frac{n}{\sigma}$$

The MLE of both α and σ can be found by resolving the above two nonlinear equations by equating $\frac{\partial \log(L)}{\partial \alpha} = 0$ and $\frac{\partial \log(L)}{\partial \sigma} = 0$, by using the numerical approach using such methods as Newton-Raphson and Nelder–Mead methods (Roussas, 2003). They numerically find minimum or maximum of an objective function in a multidimensional space. Note that the estimators $\hat{\alpha}, \hat{\sigma}^2$ are random variables and so they have their own distributions, means and variances. MLE estimate $\hat{\theta}_n$ is consistent and converges in probability to the true value as sample size n goes to infinity (Karr, 1993; Roussas, 2003).

Simulation Studies with Acceptance-Rejection Algorithm

Four simulations of the APTHN distribution are conducted using the true values $\alpha \in \{0.5, 3\}$ and $\sigma \in 1, 2$. The acceptance-rejection AR algorithm proposed by (Casella & Berger, 2002) is applied here in order to generate random observations from the new APTHN distribution. The half-normal pdf is used as a proposal density in the algorithm. Accepted samples are exact observations from the new distribution used. Iterations of 100000 are run, and accepted samples are plotted in Figure 11. The true pdfs are estimated using the histograms and density functions. The parameters are estimated from the random observations using the maximum likelihood estimation method.

The results of the four simulation studies demonstrate that the new density function has different shapes covering large intervals when larger variance compared to the smaller value. The newly included parameter α also greatly affects the shape and skewness of the density function.

7 Parameter Estimation using MLE Method

For the simulation studies, we generate samples of increasing sample sizes from the new distribution with given parameter values and estimate them back using the MLE method. The recital of the MLE is measured using the mean square error and bias for each case. Many sample sizes are considered in the experiments at sizes n =150, 300, 500, 1000, and 5000. In addition, the different values of parameters and are considered. The MLE estimates asymptotically converge in probability to the true parameter values, and the solution is unique. The results in Table 1 display estimates of MLE, MSE and bias for three different cases. The results show that as sample size increases,

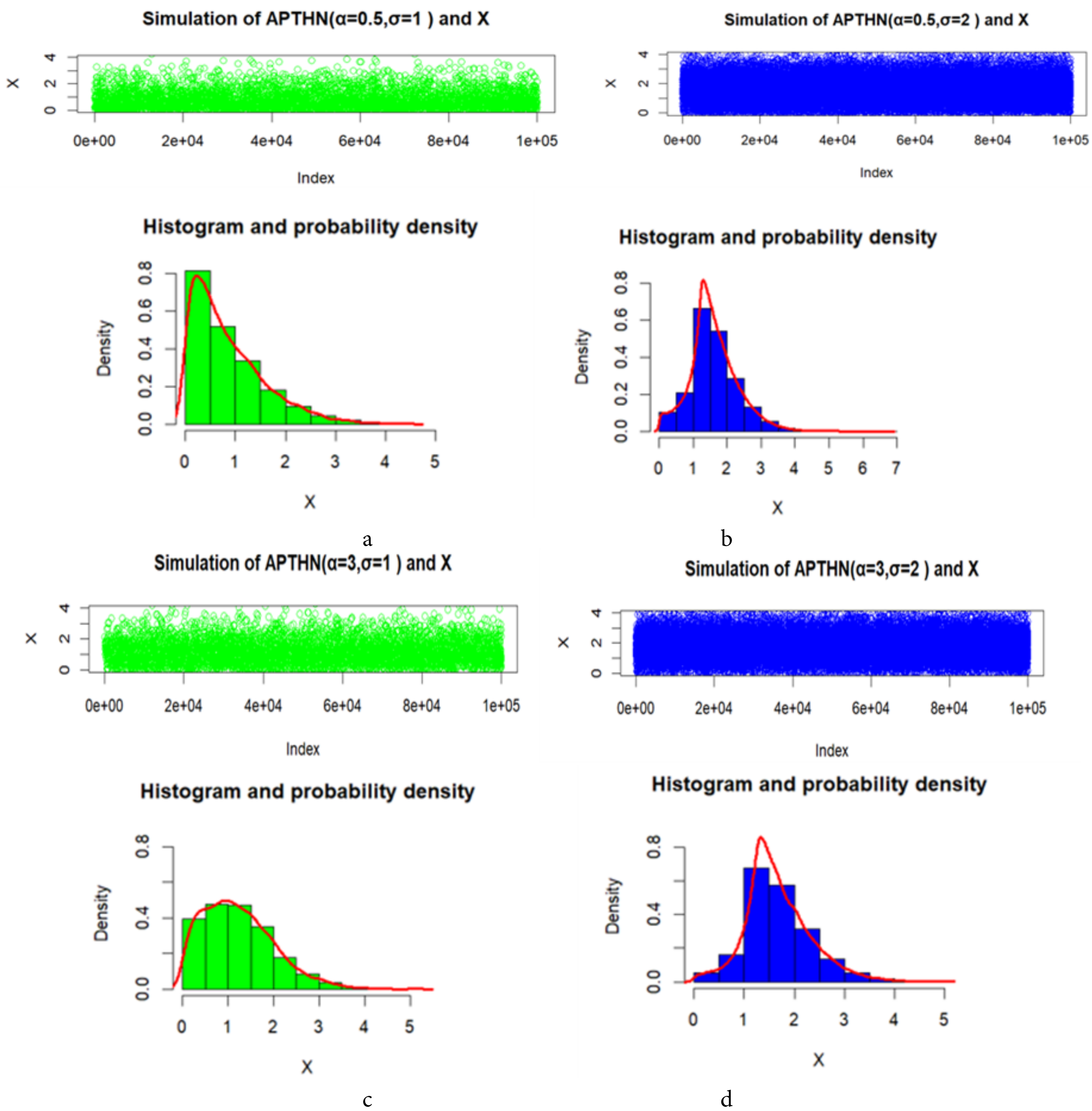


Figure 11: Plots of realization and respective histogram of the case of (a) $\alpha = 0.5\,, \sigma = 1$; (b) $\alpha = 0.5\,, \sigma = 2$; (c) $\alpha = 3\,, \sigma = 1$; (d) $\alpha = 3\,, \sigma = 2$.

the parameter estimates approach to the true value, with the respective MSE and bias-computed values both approaching zero. This holds to be true for each case of the three scenarios.

8 Model Comparison in Fitting to Three Datasets

Applications of the model to two real datasets and one simulated datasets were carried out to see the performance of the proposed APTHN distribution as compared to related one parameter existing distributions, which are half normal distribution (HND), exponential distribution (EXPD), Lindley distribution (LD), XLindley distribution (XLD), and XGamma distribution (XGD). The criteria employed are Anderson-Darling (A), Cramér-von Misses (W), Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), and HQIC (Hannan-Quinn information criterion). The distribution with the smallest values of AIC, BIC, and CAIC is considered the best model for the given data.

First, a dataset taken from Nichols and Padgett (2006) is used and it is named the breaking stress data. This dataset includes 100 carbon fiber breaking stress observations (in GPa). Several authors have utilized these data to demonstrate how various competing models can be applied. Table 2 provides the numerical outcomes of a few goodness of fit metrics. The new model APTHND has the lowest values of BIC, AIC, CAIC, and HQIC. Based on these criteria, we found that the APTHND model best fits the breaking stress data as compared to the HND, EXPD, LD, XLD, and XGD models.

Table 1: MSE and bias based on simulations from APTHN model

Parameters True Values	n	$\hat{\alpha}$	MSE ($\hat{\alpha}$)	Bias ($\hat{\alpha}$)	$\hat{\sigma}$	MSE ($\hat{\sigma}$)	Bias ($\hat{\sigma}$)
$\alpha = 8$ $\sigma = 0.3$	150	5.4402	6.5523	2.5598	0.3249	6e-04	0.0249
	300	5.8983	4.4171	2.1017	0.3108	1e-04	0.0108
	500	6.8458	1.3323	1.1542	0.3032	9e-05	0.0032
	1000	7.8323	0.0281	0.1677	0.2966	1e-05	0.0015
	5000	8.0543	0.0029	0.0543	0.2932	4e-06	0.0007
$\alpha = 6$ $\sigma = 1$	150	3.8718	4.5292	2.1282	1.2406	0.0579	0.2406
	300	5.7930	1.2076	1.2070	1.1659	0.0275	0.1659
	500	7.0989	0.4335	1.0989	1.1378	0.0190	0.1378
	1000	6.6584	0.0429	0.6584	1.1194	0.0142	0.1194
	5000	5.9841	0.0003	0.0159	1.1292	0.0037	0.0292
$\alpha = 3$ $\sigma = 1$	150	2.7260	0.2751	0.2741	1.2475	0.0612	0.2475
	300	2.7706	0.1686	0.2106	1.1781	0.0317	0.1781
	500	2.8061	0.1281	0.1361	1.1515	0.0229	0.1515
	1000	2.9231	0.0284	0.0731	1.1469	0.0216	0.1169
	5000	2.9948	0.0090	0.0147	1.0091	0.0013	0.0191

Table 2: Measures of goodness of fits for breaking stress of carbon fiber data

Distribution	Par	MLE	W	A	AIC	CAIC	BIC	HQIC	Value
APTHND	α	3.05	0.062	0.401	749.50	749.63	754.71	751.61	372.75
	σ	3.21							
HND	σ	1.19	0.059	0.401	836.10	836.14	838.71	837.15	417.05
EXPD	λ	0.8	0.214	1.119	951.15	951.19	953.75	952.20	474.57
LD	θ	0.4	0.167	0.851	977.99	978.03	980.59	979.05	487.99
XLD	θ	0.6	0.153	0.782	847.38	847.42	849.98	848.43	422.69
XGD	θ	0.15	0.133	0.692	785.75	785.79	788.36	786.81	391.88

The second dataset consider for this paper is pharmacokinetics of indomethacin data, which explains plasma concentrations of indomethacin (mcg/ml). The dataset was analyzed by authors (Davidian & Giltinan, 1995; Kwan et al., 1976; Pinheiro & Bates, 2000). Table 3 displays the results of goodness of fit criteria including W, A, AIC, CAIC, BIC, and HQIC for aircraft failure times data. The AIC and BIC value in Table 3 is relatively lower for the new distribution as compared to other distributions. The APTHN distribution performs relatively better compare to the half normal, exponential, Lindley, xLindley and xgamma distributions. The fitting plots are displayed in Figure 12.

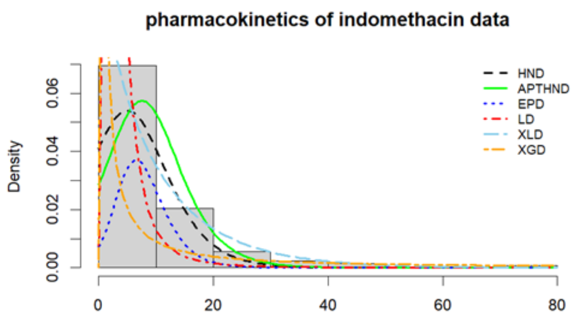


Figure 12: PPlots of fitted models to the pharmacokinetics of indomethacin data

The third dataset is one simulated data from APTHN distribution by using AR algorithm, with α=3, and σ=1. The data consists of 200 random samples. The data fitting results are given in in Table 4. The results show that the values of AIC, CAIC, BIC and HQIC are all lowest for the APTHN distribution among all models applied. The APTHN distribution still performs best.

The six distributions are fitted to three datasets including real and simulated datasets. The results show that performance of the newly suggested APTHN distribution is found to be interestingly the best. The new model is promising and so it deserves to be tested in other applications and further studies.

9 Conclusions

The purpose of this study is to develop a new continuous probability distribution using the Alpha Power Transformation (APT) method using the base Half-Normal distribution. The proposed Alpha Power Transformed Half-Normal (APTHN) distribution represents a novel contribution to statistical theory. Several statistical properties of the APTHN distribution are derived and discussed in detail. Its hazard function exhibits flexible and interesting shapes, while its probability density and cumulative distribution functions differ significantly from those of the base half-normal distribution. The performance of the proposed APTHN distribution was evaluated using one simulated dataset and two real datasets, where it outperformed six existing distributions in terms of goodness of fit. Statistical

Table 3: Measures of goodness of fits for pharmacokinetics of indomethacin data

Distribution	Par	MLE	W	A	AIC	CAIC	BIC	HQIC	Value
APTHND	α	0.19	0.263	1.692	68.91	69.10	73.29	70.64	32.45
	σ	0.8							
HND	σ	3.57	0.294	1.890	80.93	80.99	83.12	81.80	39.47
EXPD	λ	0.89	0.232	1.486	151.64	151.70	153.83	152.50	74.82
LD	θ	1.49	22.373	129.700	189.23	189.29	191.42	190.10	93.61
XLD	θ	0.87	0.239	1.538	109.15	109.21	111.34	110.02	53.58
XGD	θ	1.45	0.222	1.437	102.89	102.96	105.08	103.76	50.45

Table 4: Measures of goodness of fits for simulated data using $\alpha = 3$, and $\sigma = 1$

Distribution	Par	MLE	W	A	AIC	CAIC	BIC	HQIC	Value
APTHND	α	3.01	0.063	0.462	426.67	426.73	433.26	429.34	211.33
	σ	0.91							
HND	σ	2.08	0.031	0.259	446.4	446.5	453	449.1	221.2
EXPD	λ	0.059	0.293	1.798	1152.7	1152.8	1156.1	1154.1	575.4
LD	θ	0.053	0.095	0.747	2345.6	2345.7	2348.9	2347	1171.8
XLD	θ	0.089	0.206	1.266	1590.6	1590.7	1593.9	1592	794.3
XGD	θ	0.39	0.209	1.277	947.9	947.9	951.3	949.3	472.9

inferences including model fitting, parameter estimation, and simulation studies demonstrate that the new distribution provides valuable insights for applied probability, applied statistics, and various practical fields such as life sciences. The results of this study can serve as a foundation for future research and model development in these areas.

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Conflicts of Interest

The authors declare that there are no conflicts of interest.

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